Cambridge University Press
Fetter Lane, London

New York
Bombay, Calcutta, Madras
Toronto
Macmillan

Tokyo
Maruzen-Kabushiki-Kaisha

TRIGONOMETRY for

By

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PART IV

Infinite Series, Products, etc.

CAMBRIDGE
AT THE UNIVERSITY PRESS
1929

PRINTED IN GREAT BRITAIN

PREFACE

The study of Trigonometry at school may be divided into the following stages:

(i) A course of Numerical Trigonometry should be taken by every pupil to crown the usual mathematical work in Arithmetic, Algebra and Geometry.

(ii) A course of Algebraical Trigonometry should also be taken by all Science and Engineering pupils.

(iii) A knowledge of the elements of Complex Numbers and Series is necessary for the serious Science and Engineering students.

(iv) The study of Infinite Series and of trigonometrical functions of a Complex Variable must be taken by the Mathematical Specialists and the Physicists.

This book is divided into four parts to meet the needs of the above studies.

The book has a very large number of exercises in all parts, and the effort has been made to make these bear more directly than is usual in books on Trigonometry on the pupil's work in other branches of Mathematics and Physics—*e.g.* Mechanics, Calculus, Co-ordinate Geometry and Light. Work in three dimensions is common throughout. All exercises are grouped under appropriate headings and carefully graded, and there are sets of Miscellaneous Examples and Papers for Revision. Another feature is a large number of worked examples chosen with the special object of developing systematic work; they should be carefully studied, especially in Part I.

While Part I is suitable for the beginner, the complete book will take a pupil well up to the standard of University Scholarship Examinations, and leave him ready to read advanced treatises. No attempt has been made to tackle all the difficulties in the more advanced portions, but these have always been pointed out where

they exist, and although much is left for the pupil to amplify, it is hoped that there is nothing which he will have to forget.

An index is provided and answers to exercises.

We have to acknowledge the courtesy of H.M. Stationery Office, of the Oxford and Cambridge Joint Board, of the Oxford Local Examinations Delegacy, of the Cambridge Local Examinations Syndicate, of the University of London for permission to include exercises from public examination papers. Finally we have to thank many colleagues and friends for suggestions and criticisms.

A. W. S.
R. T. H.

HARROW,
May, 1928.

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*Articles and Exercises marked * may be omitted at a first reading.*

PART IV

CHAPTER XVII

FUNCTIONS OF COMPLEX NUMBERS

§ 1. In Chapter xv complex numbers were introduced into the Binomial Series and the Exponential Series. This chapter deals first with the circular and hyperbolic functions of complex numbers and then with exponential and logarithmic functions of complex numbers.

No attempt is made to produce a complete treatment, but only to introduce the subject, one which is full of pitfalls and difficulties.†

§ 2. Circular Functions of Complex Variables.

When a function has been defined for a limited range of values and we wish to define it for an extended range, we usually take some property of the function that has been proved for the limited range and try whether that will furnish us with a definition for the extended range. (Remember how the meaning of 10^x was extended.) In such cases it is important to see that the new definition over the extended range includes the functions for the limited range, as particular cases.

In Chap. xv, § 4, we saw that, when x is real,

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \dots \qquad \sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \dots$$

$$= \frac{\exp(ix) + \exp(-ix)}{2}, \qquad = \frac{\exp(ix) - \exp(-ix)}{2i}.$$

If z is a complex number, $\cos z$ and $\sin z$ have at present no meaning, but we shall take the above properties and define $\sin z$ and $\cos z$ as follows:

Def. If z is a complex number,

$$\cos z = \frac{\exp(iz) + \exp(-iz)}{2}, \qquad \sin z = \frac{\exp(iz) - \exp(-iz)}{2i}.$$

These forms are, of course, merely shorthand for two infinite series.

Now a real number x is a particular case of a complex number $x + iy$, so that the above definition defines $\cos x$ and $\sin x$, where x is real.

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† The student should pursue the subject in Hobson's *Trigonometry*, Hardy's *Pure Mathematics*, or Chrystal's *Algebra*.

Further, in Chap. XV, §5, we pointed out that, if we defined $\cos x$ and $\sin x$ by means of these series, we could deduce all the analytical results of Chaps. VI to VIII. Hence we see that our new definition does not contradict the old definition.

From this definition we can deduce the relations

$$\cos(-z) = \cos z, \quad \sin(-z) = -\sin z, \quad \text{............}(1)$$

$$\cos^2 z + \sin^2 z = 1, \quad \text{........................}(2)$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2, \text{............}(3)$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2. \text{............}(4)$$

E.g.
$$\sin(z_1 + z_2) = \frac{\exp(i\,\overline{z_1 + z_2}) - \exp(-i\,\overline{z_1 + z_2})}{2i}$$

$$= \frac{\exp(iz_1)\exp(iz_2) - \exp(-iz_1)\exp(-iz_2)}{2i}$$

$$= \frac{[\exp(iz_1) - \exp(-iz_1)]}{2i}\frac{[\exp(iz_2) + \exp(-iz_2)]}{2}$$

$$+ \frac{[\exp(iz_1) + \exp(-iz_1)]}{2}\frac{[\exp(iz_2) - \exp(-iz_2)]}{2i}$$

$$= \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

As the formulae (1) to (4) above are now seen to be true for complex values of z as well as for real values, it follows that the formulae of Chaps. VI to VIII which were deduced from formulae (1) to (4) for real values are also true for complex values.

Also as $\cos(z + 2n\pi) = \cos z \cos 2n\pi - \sin z \sin 2n\pi = \cos z$, and similarly $\sin(z + 2n\pi) = \sin z$, etc., it follows that the circular functions of a complex variable have the same real periods as the corresponding circular functions of a real variable.

§3. Hyperbolic Functions of Complex Variables.

Just as we have defined cos and sin for a complex variable by means of series, so we define cosh and sinh for a complex variable.

Def. If z is complex,

$$\cosh z = \frac{\exp(z) + \exp(-z)}{2}, \qquad \sinh z = \frac{\exp(z) - \exp(-z)}{2}.$$

This definition obviously includes the definition for a real variable, and the formulae deduced in Chap. XV, §7 for hyperbolic functions of a real variable also hold for a complex variable.

§ 4. Connection between Circular and Hyperbolic Functions.

Since
$$\cosh z = \frac{\exp(z) + \exp(-z)}{2}$$
$$= 1 + \frac{z^2}{\lfloor 2} + \frac{z^4}{\lfloor 4} + \ldots$$
$$= 1 - \frac{(iz)^2}{\lfloor 2} + \frac{(iz)^4}{\lfloor 4} + \ldots.$$
$$\therefore \ \cosh z = \cos(iz).$$

Similarly $\sinh z = \frac{1}{i}\sin(iz)$; hence we have

$$\sin iz = i \sinh z, \quad \cos iz = \cosh z, \quad \tan iz = i \tanh z,$$
$$\sinh iz = i \sin z, \quad \cosh iz = \cos z, \quad \tanh iz = i \tan z.$$

From these relations it is possible to deduce formulae in hyperbolic functions from formulae in circular functions.

These relations enable us to see that $\cosh z$ and $\sinh z$ have an imaginary period $2\pi i$, and $\tanh z$ an imaginary period πi, *e.g.*
$$\cosh z = \cos iz = \cos(-2n\pi + iz)$$
$$= \cos i\,(2n\pi i + z)$$
$$= \cosh(2n\pi i + z).$$

See also § 8.

§ 5. Example i. *Write $\tan(x+iy)$ in the form $u+iv$, where x, y, u, v are real.*

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}$$
$$= \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$
$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy}$$
$$= \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}.$$
$$\therefore \quad u = \frac{\sin 2x}{\cos 2x + \cosh 2y}, \quad v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}.$$

Example ii. *If* $\tan(x+iy)=u+iv$ *where* x, y, u *and* v *are real, prove that* $u^2+v^2+2u\cot 2x=1$.

$$\tan(x+iy)=u+iv.$$

$$\therefore\quad \tan(x-iy)=u-iv,$$

$$\therefore\quad \tan\{\overline{x+iy}+\overline{x-iy}\}=\frac{(u+iv)+(u-iv)}{1-(u+iv)(u-iv)},$$

$$i.e.\quad \tan 2x=\frac{2u}{1-(u^2+v^2)}.$$

$$\therefore\quad u^2+v^2+2u\cot 2x=1.$$

Example iii. *If* $\sin(\mathrm{A}+i\mathrm{B})=\cos\theta+i\sin\theta$ *where* A, B, θ *are real, prove that* $\cos^2\mathrm{A}=\sinh^2\mathrm{B}$.

$$\sin(\mathrm{A}+i\mathrm{B})=\cos\theta+i\sin\theta.$$

$$\therefore\quad \sin\mathrm{A}\cos i\mathrm{B}+\cos\mathrm{A}\sin i\mathrm{B}=\cos\theta+i\sin\theta,$$

$$\therefore\quad \sin\mathrm{A}\cosh\mathrm{B}+i\cos\mathrm{A}\sinh\mathrm{B}=\cos\theta+i\sin\theta.$$

$$\therefore\quad \sin\mathrm{A}\cosh\mathrm{B}=\cos\theta,$$

$$\cos\mathrm{A}\sinh\mathrm{B}=\sin\theta.$$

$$\therefore\quad \sin^2\mathrm{A}\cosh^2\mathrm{B}+\cos^2\mathrm{A}\sinh^2\mathrm{B}=1,$$

$$\therefore\quad (1-\cos^2\mathrm{A})(1+\sinh^2\mathrm{B})+\cos^2\mathrm{A}\sinh^2\mathrm{B}=1.$$

$$\therefore\quad \sinh^2\mathrm{B}-\cos^2\mathrm{A}=0.$$

EXERCISE XVII. a.

[*All letters represent real numbers unless otherwise stated.*]

1. Prove, from the definitions of § 2, that if z is a complex number:

(i) $\sin 2z=2\sin z\cos z$,

(ii) $\cos 2z=\cos^2 z-\sin^2 z=2\cos^2 z-1=1-2\sin^2 z$,

(iii) $\cos(z_1-z_2)=\cos z_1\cos z_2+\sin z_1\sin z_2$.

2. Prove:

(i) $\cosh\left(x+i\frac{\pi}{2}\right)=i\sinh x$, (iv) $\cosh\left(\frac{i\pi}{2}-x\right)=-i\sinh x$,

(ii) $\sinh\left(x+i\frac{\pi}{2}\right)=i\cosh x$, (v) $\sinh\left(\frac{i\pi}{2}-x\right)=i\cosh x$,

(iii) $\tanh\left(x+i\frac{\pi}{2}\right)=\coth x$, (vi) $\tanh\left(\frac{i\pi}{2}-x\right)=-\coth x$.

3. Deduce formulae for the following, by the substitutions of § 4, from the analogous formulae in circular functions of real numbers:

(i) $\cosh(u+v)$, (iii) $\cosh 2u$, (v) $\sinh 2u+\sinh 2v$.

(ii) $\sinh 2u$, (iv) $\tanh 3u$,

4. Write in the form $u+iv$ (*i.e.* separate into real and imaginary parts):

(i) $\sin(x+iy)$, (iv) $\sinh(x+iy)$, (vii) $\tanh(x+iy)$.

(ii) $\cos(x-iy)$, (v) $\sec(x+iy)$,

(iii) $\cosh(x+iy)$, (vi) $\operatorname{cosech}(x+iy)$,

5. If
$$\sin(\alpha+i\beta)=\cos\theta+i\sin\theta,$$
prove that
$$\sin\theta=\pm\cos^2\alpha=\pm\sinh^2\beta.$$

6. If
$$\sin(x+iy)=\tan(u+iv),$$
show that
$$\sin 2u\tanh y=\sinh 2v\tan x.$$

7. If
$$\sin(\alpha+i\beta)=x+iy=(r,\theta),$$
prove that

$$\text{(i)}\quad \frac{x^2}{\sin^2\alpha}-\frac{y^2}{\cos^2\alpha}=1,\quad \frac{x^2}{\cosh^2\beta}+\frac{y^2}{\sinh^2\beta}=1,$$

$$\text{(ii)}\quad \tan\theta=\tanh\beta\cot\alpha\ \text{ and }\ r^2=\tfrac{1}{2}(\cosh 2\beta-\cos 2\alpha).$$

8. If α is acute and if
$$\tan(\theta+i\phi)=\cos\alpha+i\sin\alpha,$$
prove that
$$4\theta=(2n+1)\pi\ \text{ and }\ \cosh 2\phi=\sec\alpha.$$

9. If
$$\tan(x+iy)=u+iv,$$
prove that
$$\frac{u}{v}=\frac{\sin 2x}{\sinh 2y}\ \text{ and }\ \frac{1-u^2-v^2}{1+u^2+v^2}=\frac{\cos 2x}{\cosh 2y}.$$

10. If
$$x+iy=\tan\tfrac{1}{2}(u+iv),$$
then
$$x^2+y^2=1-2x\cot u=-1+2y\coth v,$$
and show that u is the angle between the lines joining the point (x, y) to the points $(0, \pm 1)$.

11. If
$$\tanh u=\frac{\sin x}{\cosh y}\ \text{ and }\ \tan v=\frac{\sinh y}{\cos x},$$
prove that
$$\tanh(u+iv)=\sin(x+iy).$$

12. If
$$\cos(\alpha+i\beta)=(r,\theta),$$
prove that
$$\beta=\tfrac{1}{2}\log\frac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}.$$

13. If
$$\tan(\alpha+i\beta)=\tan\phi+i\sec\phi,$$
where α, β, ϕ are real, prove that
$$2\alpha=n\pi+\frac{\pi}{2}+\phi\ \text{ and }\ 4\beta=\log\left(\frac{1+\cos\phi}{1-\cos\phi}\right),$$
where n is any integer.

14. If
$$\cosh(x+iy)=\cot(u+iv),$$
prove that
$$\sinh 2v\operatorname{cosec} 2u+\tanh x\tan y=0,$$
and that
$$\coth 2v=-\{\cosh 2x+\cos 2y+2\}/4\sinh x\sin y.$$

§ 6. Inverse Circular Functions.

If $$\cos(x+iy) = u + iv,$$

then $x+iy$ is defined as *an* inverse cosine of $u+iv$.

But $\cos(x+iy) = \cos(2n\pi \pm \overline{x+iy})$ by § 2, so that $2n\pi \pm (x+iy)$ is in general an inverse cosine of $u+iv$ for any integral, or zero, value of n.

Hence we say that

$$\text{Cos}^{-1}(u+iv) = 2n\pi \pm (x+iy) \text{ or } 2n\pi \pm \cos^{-1}(u+iv),$$

where Cos is written instead of cos, to indicate that the many-valuedness of the function is being considered.

We shall in future use $\cos^{-1}(u+iv)$ to represent the principal value (defined below) of $\text{Cos}^{-1}(u+iv)$.

The **principal value** of $\text{Cos}^{-1}(u+iv)$ is that value for which the real part lies between 0 and π, or $=0$ or π.

Note that this definition agrees with that for the principal value of $\cos^{-1} a$, where a is real. See Chap. v, § 5, p. 86.

Similarly if $$\sin(x+iy) = u+iv,$$

$$\text{Sin}^{-1}(u+iv) = n\pi + (-1)^n(x+iy) \text{ or } n\pi + (-1)^n \sin^{-1}(u+iv),$$

and if $$\tan(x+iy) = u+iv,$$

$$\text{Tan}^{-1}(u+iv) = n\pi + (x+iy) \text{ or } n\pi + \tan^{-1}(u+iv).$$

The principal values of $\text{Sin}^{-1}(u+iv)$ and $\text{Tan}^{-1}(u+iv)$ are in each case those values for which the real part lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, or $=-\frac{\pi}{2}$ or $+\frac{\pi}{2}$. See Chap. v, § 5, p. 86.

Inverse Hyperbolic Functions.

From § 4, it follows that

$$\text{Cosh}^{-1}(u+iv) = 2n\pi i \pm \cosh^{-1}(u+iv),$$
$$\text{Sinh}^{-1}(u+iv) = n\pi i + (-1)^n \sinh^{-1}(u+iv),$$
$$\text{Tanh}^{-1}(u+iv) = n\pi i + \tanh^{-1}(u+iv).$$

§ 7. **Example iv.** *Write $\cos^{-1}(\cos\theta+i\sin\theta)$ in the form $x+iy$, where θ is a positive angle $<\pi$.*

Let $$\cos^{-1}(\cos\theta+i\sin\theta)=x+iy.$$

$$\therefore\ \cos\theta+i\sin\theta=\cos(x+iy)$$
$$=\cos x\cosh y-i\sin x\sinh y, \quad\text{.........(i)}$$
$$\therefore\ \cos x\cosh y=\cos\theta, \quad\text{.........(ii)}$$
$$\sin x\sinh y=-\sin\theta. \quad\text{.........(iii)}$$
$$\therefore\ \text{(iii)},\ (1-\cos^2 x)(\cosh^2 y-1)=\sin^2\theta,$$
$$\therefore\ \cosh^2 y+\cos^2 x-1-\cos^2 x\cosh^2 y=\sin^2\theta.$$
$$\therefore\ \text{(ii)},\ \cosh^2 y+\cos^2 x=1+\cos^2\theta+\sin^2\theta=2,$$
$$\therefore\ \text{(ii)},\ (\cosh y+\cos x)^2=2+2\cos\theta=4\cos^2\frac{\theta}{2}.$$
$$\therefore\ \cosh y+\cos x=+2\cos\frac{\theta}{2}\ (\text{as }\cosh y>1,\ \theta\text{ being }+\text{ and }<\pi).$$

Similarly $\cosh y-\cos x=+2\sin\dfrac{\theta}{2}$ (as $\cosh y>1$).

$$\therefore\ \cosh y=\cos\frac{\theta}{2}+\sin\frac{\theta}{2}, \quad\text{.........(iv)}$$

and $$\cos x=\cos\frac{\theta}{2}-\sin\frac{\theta}{2}. \quad\text{.........(v)}$$

$\therefore\ x=\cos^{-1}\left(\cos\dfrac{\theta}{2}-\sin\dfrac{\theta}{2}\right)$, since the principal value of x corresponds to the principal value of $\cos^{-1}(\cos\theta+i\sin\theta)$, and

$$y=\pm\cosh^{-1}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right).$$

Now x and θ each lie between 0 and π, so $\sin x$ and $\sin\theta$ are both positive.

$$\therefore\ \text{by (iii)}\ y=-\cosh^{-1}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right),$$

$$\therefore\ \cos^{-1}(\cos\theta+i\sin\theta)=\cos^{-1}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)-i\cosh^{-1}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)$$

These results may be thrown into a neater form thus:

$$\text{(iv)}\ 1+\sinh^2 y=1+\sin\theta,$$
$$\text{(v)}\ 1-\sin^2 x=1-\sin\theta.$$
$$\therefore\ x=\sin^{-1}(\pm\sqrt{\sin\theta}),\quad y=\sinh^{-1}(\pm\sqrt{\sin\theta}).$$

Also the principal value of x must lie between 0 and π.

$$\therefore\ x=\sin^{-1}(+\sqrt{\sin\theta}),$$

and by (iii) $$y=-\sinh^{-1}(\sqrt{\sin\theta}).$$

Example v. *Write $\tan^{-1}(\alpha+i\beta)$ in the form $x+iy$, where α, β, x, y are real.*

Let
$$\tan^{-1}(\alpha+i\beta)=x+iy.$$
$$\therefore\ \tan(x+iy)=\alpha+i\beta.$$

But
$$\tan(x+iy)=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y},\ \text{from § 5, Ex. i.}$$

$$\therefore\ \alpha=\frac{\sin 2x}{\cos 2x+\cosh 2y},\quad \beta=\frac{\sinh 2y}{\cos 2x+\cosh 2y}. \quad \text{............(i)}$$

Now $\tan^{-1}(\alpha+i\beta)$ is the principal value of the function, therefore x, its real part, if positive, lies between 0 and $\frac{\pi}{2}$; and, if negative, between 0 and $-\frac{\pi}{2}$. Therefore from (i) we see that α and x always have the same sign, as the denominator is bound to be positive ($\cosh 2y>1>\cos 2x$); similarly β and y always have the same sign.

Also
$$\alpha^2+\beta^2=\frac{\sin^2 2x+\sinh^2 2y}{(\cos 2x+\cosh 2y)^2}=\frac{\cosh^2 2y-\cos^2 2x}{(\cos 2x+\cosh 2y)^2}$$
$$=\frac{\cosh 2y-\cos 2x}{\cosh 2y+\cos 2x}.$$

$$\therefore\ 1-\alpha^2-\beta^2=\frac{2\cos 2x}{\cosh 2y+\cos 2x}, \quad \text{............(ii)}$$

$$1+\alpha^2+\beta^2=\frac{2\cosh 2y}{\cosh 2y+\cos 2x}. \quad \text{............(iii)}$$

$$\therefore\ \text{(i), (iii)},\ \tanh 2y=\frac{2\beta}{1+\alpha^2+\beta^2},\quad \therefore\ y=\tfrac{1}{2}\tanh^{-1}\frac{2\beta}{1+\alpha^2+\beta^2},$$

and (i), (ii),
$$\tan 2x=\frac{2\alpha}{1-\alpha^2-\beta^2},\quad \therefore\ 2x=n\pi+\tan^{-1}\frac{2\alpha}{1-\alpha^2-\beta^2}. \quad \text{.........(iv)}$$

$$\therefore\ x=\frac{n\pi}{2}+\tfrac{1}{2}\tan^{-1}\frac{2\alpha}{1-\alpha^2-\beta^2}.$$

Two cases now arise.

If $\alpha^2+\beta^2<1$, n being taken such that x has the same sign as α, n must be zero for the principal value.

If $\alpha^2+\beta^2>1$, for x to have the same sign as α, n must be $+1$ if α is positive, and -1 if α is negative.

Hence
$$\tan^{-1}(\alpha+i\beta)=\frac{n\pi}{2}+\tfrac{1}{2}\tan^{-1}\frac{2\alpha}{1-\alpha^2-\beta^2}+i\,\tfrac{1}{2}\tanh^{-1}\frac{2\beta}{1+\alpha^2+\beta^2},$$
where n has the value 0, ± 1, according to the conditions above stated.

If $\alpha^2+\beta^2=1$ from (iv),
$$\tan^{-1}(\alpha+i\beta)=\frac{\pi}{4}+i\,\tfrac{1}{2}\tanh^{-1}\beta.$$

EXERCISE XVII. b.

[*All letters represent real numbers unless otherwise stated.*]

1. Prove that, if $A+iB=\sin^{-1}(\cos\theta+i\sin\theta)$, then $\cos^2 A=\sinh^2 B$.

2. If $\cos^{-1}(a+i\beta)=A+iB$, prove that
(i) $a^2\sec^2 A-\beta^2\operatorname{cosec}^2 A=1$, (ii) $a^2\operatorname{sech}^2 B+\beta^2\operatorname{cosech}^2 B=1$.

3. Prove that one value of $\sin^{-1}\left\{\dfrac{5\sqrt{7}+9i}{16}\right\}$ is $\cos^{-1}\frac{3}{4}+i\log 2$.

4. Show that one value of $\tan^{-1}\dfrac{x+iy}{x-iy}$ is equal to one value of $\dfrac{\pi}{4}+\dfrac{i}{2}\log\dfrac{x+y}{x-y}$.

5. Show that, if

$$\cosh^{-1}(x+iy)+\cosh^{-1}(x-iy)=\cosh^{-1}a,$$

then

$$2(a-1)x^2+2(a+1)y^2=a^2-1.$$

6. If

$$\tan^{-1}(\xi+i\eta)=\sin^{-1}(x+iy),$$

prove that

$$\xi^2+\eta^2=(x^2+y^2)/\sqrt{x^4+y^4+2x^2y^2-2x^2+2y^2+1}.$$

---

*7. If

$$\operatorname{Sin}^{-1}2=a+i\beta,$$

show that

$$a=2n\pi+\frac{\pi}{2} \text{ and } \beta=\log(2\pm\sqrt{3}),$$

where n is any integer.

*8. Prove that

$$\operatorname{Cos}^{-1}1{\cdot}5=2n\pi+i\log\left(\frac{3\pm\sqrt{5}}{2}\right).$$

*9. Prove that

$$\operatorname{Sec}^{-1}0{\cdot}5=2n\pi+i\log(2\pm\sqrt{3}).$$

*10. Write $\sin^{-1}(\cos\theta+i\sin\theta)$ in the form $x+iy$, where θ is a positive angle $<\pi$.

*11. Write $\cos^{-1}(a+i\beta)$ in the form $x+iy$.

*12. Prove that $\operatorname{Tan}^{-1}(\cos\theta+i\sin\theta)$ is equal to

$$n\pi+\frac{\pi}{4}+i\,\tfrac{1}{2}\tanh^{-1}(\sin\theta), \text{ when } \cos\theta \text{ is positive,}$$

and

$$n\pi-\frac{\pi}{4}+i\,\tfrac{1}{2}\tanh^{-1}(\sin\theta), \text{ when } \cos\theta \text{ is negative.}$$

*13. Prove that, when a lies between 0 and π,

$$\operatorname{Sin}^{-1}(\operatorname{cosec}a)=2n\pi+\frac{\pi}{2}\pm i\log\tan\frac{a}{2}.$$

***14.** Prove that

$$\text{Cos}^{-1}(\sec a) = 2n\pi \pm i \log(\tan a + \sec a), \text{ when } \sec a \text{ is positive,}$$

and $$(2n+1)\pi \pm i \log(\tan a - \sec a), \text{ when } \sec a \text{ is negative.}$$

***15.** Prove that

$$\text{Coth}^{-1}(x+iy) = \tfrac{1}{4} \log \left\{\frac{(x+1)^2+y^2}{(x-1)^2+y^2}\right\} + i\left\{n\pi + \tfrac{1}{2}\tan^{-1}\frac{2y}{1-x^2-y^2}\right\}.$$

***16.** Show that the general solution of $a\cos\theta + b\sin\theta = c$ when $c > \sqrt{a^2+b^2}$ is

$$\theta = 2n\pi + \tan^{-1}\frac{b}{a} \pm i\cosh^{-1}\frac{c}{\sqrt{a^2+b^2}},$$

provided the value assigned to $\tan^{-1}\dfrac{b}{a}$ is also a value of $\cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}}$.

Exponential and Logarithmic Functions.

§8. To show that the exponential function of a complex number is periodic, the period being $2\pi i$.

We have already seen in §4 that the hyperbolic functions, which are really exponential functions, have imaginary periods.

In Chap. xv, §2, we defined the exponential function thus:

$$\exp(z) = 1 + \frac{z}{\underline{|1}} + \frac{z^2}{\underline{|2}} + \dots,$$

and we showed that, if $z = x + iy$, where x and y are real,

$$\begin{aligned}\exp(z) &= \exp(x+iy)\\ &= \exp(x)\exp(iy)\\ &= \exp(x)(\cos y + i\sin y),\end{aligned}$$

$$\therefore \exp(z) = \exp(x)\{\cos(y+2n\pi) + i\sin(y+2n\pi)\},$$

where n is any integer,

$$\begin{aligned}&= \exp(x)\exp(i.\overline{y+2n\pi})\\ &= \exp(x+iy+i.2n\pi)\\ &= \exp(z+2n\pi i).\end{aligned}$$

$\therefore$ $\exp(z)$ has an imaginary period $2\pi i$.

§9. *Def.* **If $u = \exp(z)$, z is defined to be the logarithm of u, or in symbols $z = \text{Log } u$.**

If z is real, we see that this definition agrees with the ordinary definition of the logarithm of a real number to the base e, *i.e.* the Naperian logarithm of the number.

Since, by § 8, $\exp(z) = \exp(z + 2n\pi i)$, it follows that $z + 2n\pi i$, the logarithm of u, is a many-valued function; this we denote by writing $\text{Log}\, u$ for the general value.

If $\log u$ denotes the principal value (defined in § 10 below) of $\text{Log}\, u$, since

$$u = \exp(z) = \exp(z + 2n\pi i),$$

$$\therefore \quad \text{Log}\, u = \log u + 2n\pi i.$$

§ 10. To show that $\text{Log}\,(r, \theta) = \log r + i\,(\theta + 2n\pi)$.

$$\begin{aligned}
\text{Let} \qquad & \text{Log}\,(r, \theta) = u + iv. \\
\therefore \quad & (r, \theta) = \exp(u + iv) \\
& \qquad = \exp(u) \times \exp(iv), \\
\therefore \quad & r(\cos\theta + i\sin\theta) = \exp(u)\,(\cos v + i \sin v), \\
\therefore \quad & \exp(u) = r \text{ and } v = \theta + 2n\pi, \\
\therefore \quad & u = \log r. \\
\therefore \quad & \text{Log}\,(r, \theta) = \log r + i\,(\theta + 2n\pi).
\end{aligned}$$

Def. $\log(r, \theta)$, the **principal value** of $\text{Log}\,(r, \theta)$, is the value of $\log r + i\,(\theta + 2n\pi)$ for which the angle lies between $-\pi$ and $+\pi$, the limits including $+\pi$ but excluding $-\pi$.

If $x + iy = (r, \theta)$, $\tan^{-1}\dfrac{y}{x}$, which represents the principal value of the angle, has a range from $-\dfrac{\pi}{2}$ to $+\dfrac{\pi}{2}$ (see Chap. v, § 5, p. 86), whereas θ has a range from $-\pi$ to $+\pi$; it follows that

$$\text{Log}\,(x + iy) = \log\sqrt{x^2 + y^2} + i\left(\tan^{-1}\frac{y}{x} + 2n\pi\right),$$

when x is positive or zero, and

$$\text{Log}\,(x + iy) = \log\sqrt{x^2 + y^2} + i\left(\tan^{-1}\frac{y}{x} + \overline{2n+1}\pi\right),$$

when x is negative.

It should be clear that, if z_1 and z_2 are any two complex numbers,

$$\text{Log}\, z_1 z_2 = \text{Log}\, z_1 + \text{Log}\, z_2,$$

but the following cases arise when principal values only are considered.

Let α, β be the principal values of the amplitudes of z_1, z_2, so that each lies between $-\pi$ and $+\pi$, then

$$\log z_1z_2 = \log z_1 + \log z_2 - 2\pi i, \text{ when } \alpha+\beta > \pi,$$
$$\log z_1z_2 = \log z_1 + \log z_2, \text{ when } \alpha+\beta \text{ lies between } \pm\pi, \text{ or } = \pi,$$
$$\log z_1z_2 = \log z_1 + \log z_2 + 2\pi i, \text{ when } \alpha+\beta = \text{ or } < -\pi.$$

Similar results hold for $\log \dfrac{z_1}{z_2}$.

§ 11. Complex Indices.

Def. If a and z are complex numbers, a^z is defined to be

$$\exp(z \operatorname{Log} a).$$

From the definition we see that, as $\operatorname{Log} a$ is many-valued, $\exp(z \operatorname{Log} a)$ is many-valued, so that a^z is many-valued.

We shall use the definition in some examples, but shall not pursue the subject further. We must, however, point out the danger of treating a many-valued function as though it were a single-valued function.

§ 12. **Example vi.** *Find the values of Log* $(1+i)$.

$$1+i = \left(\sqrt{2},\ 2n\pi + \frac{\pi}{4}\right).$$

$$\therefore\ \operatorname{Log}(1+i) = \log\sqrt{2} + i\left(2n\pi + \frac{\pi}{4}\right) \text{ where } n \text{ is any integer,}$$
$$= \tfrac{1}{2}\log 2 + i\left(2n\pi + \frac{\pi}{4}\right).$$

Note. Hence the values of $\operatorname{Log}(1+i)$ can be represented geometrically by points which all lie on a straight line perpendicular to the real axis.

Example vii. *Evaluate Log* (-1).

$$-1 = (1,\ 2n\pi + \pi).$$

$$\therefore\ \operatorname{Log}(-1) = \log 1 + i(2n\pi + \pi) \text{ where } n \text{ is any integer,}$$
$$= (2n+1)\pi i.$$

Example viii. *Resolve* $\operatorname{Log}\cos(x+iy)$ *into its real and imaginary parts, where x and y are real.*

Let $$\operatorname{Log}\cos(x+iy) = u + iv.$$

$$\therefore\ \exp(u+iv) = \cos(x+iy)$$
$$= \cos x \cosh y - i \sin x \sinh y,$$

$\therefore\quad \exp(u)(\cos v + i\sin v) = r\{\cos(\theta+2n\pi) - i\sin(\theta+2n\pi)\}$, say,

where $\quad r^2 = \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y$

$$= \tfrac{1}{4}[(1+\cos 2x)(1+\cosh 2y) + (1-\cos 2x)(\cosh 2y - 1)]$$

$$= \tfrac{1}{2}(\cos 2x + \cosh 2y),$$

and $\quad \theta = \tan^{-1}(\tan x\tanh y)$, if $\cos x$ is positive ($\cosh x$ must be positive),

or $\quad\quad = \pi + \tan^{-1}(\tan x\tanh y)$, if $\cos x$ is negative.

$$\therefore\quad \operatorname{Log}\cos(x+iy) = \tfrac{1}{2}\log\frac{\cos 2x + \cosh 2y}{2} - i(2n\pi+\theta),$$

where θ has the value stated above.

Example ix. *Find the values of a^i.*

By definition

$$a^i = \exp(i\operatorname{Log} a)$$
$$= \exp i(\log a + 2n\pi i) \quad \text{where } n \text{ is any integer,}$$
$$= \exp(2n\pi)\times\exp(i\log a)$$
$$= \exp(2n\pi)\,.\,\{\cos(\log a) + i\sin(\log a)\}.$$

Example x. *If $(x+iy)^{\alpha+i\beta}$ is expressed in the form $X+iY$, show that one of the values is real if $\frac{1}{2}\beta\log(x^2+y^2)+\alpha\tan^{-1}\frac{y}{x}$ is a multiple of π.*

Let $x+iy=(r,\theta)$ and suppose x is positive.

$$(x+iy)^{\alpha+i\beta} = \exp\{(\alpha+i\beta)\log(x+iy)\} \quad \text{by def.,}$$
$$= \exp[(\alpha+i\beta)\{\log r + i(\theta+2n\pi)\}]$$
$$= \exp[\{\alpha\log r - \beta(\theta+2n\pi)\} + i\{\beta\log r + \alpha(\theta+2n\pi)\}]$$
$$= \exp\{\alpha\log r - \beta(\theta+2n\pi)\}\,.\,[\cos\{\beta\log r + \alpha(\theta+2n\pi)\}$$
$$+ i\sin\{\beta\log r + \alpha(\theta+2n\pi)\}].$$

Hence one value will be real if

$$\beta\log r + \alpha(\theta+2n\pi) \text{ is a multiple of } \pi,$$

i.e. if $\quad \beta\log r + \alpha\theta$ is a multiple of π,

i.e. if $\quad \frac{1}{2}\beta\log(x^2+y^2) + \alpha\tan^{-1}\frac{y}{x}$ is a multiple of π.

If x is negative for $2n\pi$ we must take $(2n+1)\pi$ and the same result is true.

EXERCISE XVII. c.

[*All letters represent real numbers unless otherwise stated.*]

1. Write in the form $x+iy$:

(i) $\operatorname{Log} 1$, (ii) $\operatorname{Log}(-a)$, (iii) $\operatorname{Log} i$.

2. Prove that, if θ lies between $\pm\frac{\pi}{2}$,

$$\log(1+i\tan\theta) = \log\sec\theta + i\theta.$$

3. Prove that $\log \dfrac{x+iy}{x-iy}$ is a value of $2i\,\mathrm{Tan}^{-1}\dfrac{y}{x}$.

4. Prove that $\log \tan\left(\dfrac{\pi}{4}+i\dfrac{\theta}{2}\right)=i\tan^{-1}(\sinh\theta)$.

5. Prove that $\log \dfrac{\cos(x-iy)}{\cos(x+iy)}=2i\tan^{-1}(\tan x\tanh y)$.

6. Prove that

$$\log(1+\cos 2\theta+i\sin 2\theta)=\log(2\cos\theta)+i\theta,$$

provided θ lies between $\pm\dfrac{\pi}{2}$.

7. If $\log\cos(\theta-\phi i)=\mathrm{A}+\mathrm{B}i$, prove that

$$\mathrm{A}=\tfrac{1}{2}\log\tfrac{1}{2}(\cos 2\theta+\cosh 2\phi),$$

and also that $\quad \phi=\tfrac{1}{2}\{\log\sin(\theta+\mathrm{B})-\log\sin(\theta-\mathrm{B})\}$.

8. Resolve $\mathrm{Log}\sin(x+iy)$ into real and imaginary parts.

---

9. Prove that

$$i^x=\cos\left\{(4n+1)\frac{\pi}{2}x\right\}+i\sin\left\{(4n+1)\frac{\pi}{2}x\right\},\text{ where } n \text{ is an integer.}$$

10. Express $(x+iy)^i$ in the form $\mathrm{A}+i\mathrm{B}$.

11. Reduce to the form (r,θ), $i^{\frac{1}{3}}$ and $\sqrt{1+\sqrt{i}}$.

12. If $i^{x+iy}=x+iy$, prove that $x^2+y^2=\exp\{-(4n+1)\pi y\}$.

13. If $(a+ib)^{c+id}$ is wholly real and principal values only are considered, prove

$$(a+ib)^{c+id}=\sqrt[2c]{(a^2+b^2)^{(c^2+d^2)}}.$$

14. Prove that, if principal values only are considered, the real part of $i^{\log(1+i)}$ is $e^{-\frac{\pi^2}{8}}\cos\left(\dfrac{\pi}{4}\log 2\right)$.

15. If $\dfrac{(1+i)^{x+iy}}{(1-i)^{x-iy}}=(r,\theta)$, then one value of θ is $\tfrac{1}{2}x\pi+y\log_e 2$.

16. Prove that, if $\log\{\log(x+iy)\}=p+iq$, then

$$y=x\tan\{\tan q\log\sqrt{x^2+y^2}\}.$$

17. If principal values only are considered, prove that, if

$$[\cos(\alpha-i\beta)]^{p+iq}=\mathrm{A}+i\mathrm{B},$$

$$\tan^{-1}\frac{\mathrm{B}}{\mathrm{A}}=\frac{q}{2}\log(\cosh^2\beta-\sin^2\alpha)+p\tan^{-1}(\tan\alpha\tanh\beta).$$

For further examples see Miscellaneous Exercises IV, p. 391.

CHAPTER XVIII

INFINITE SERIES

"*The theory of series is both difficult and incomplete; but the difficulty is not of the kind which the student perceives, and the deficiency is also unseen, because, in fact, the imperfect theory which is first presented to him is more than sufficient for all the series of which he has any experience. He grows, therefore, in the conviction, that whatever series may be proposed, or may occur, the theory may always be made satisfactory.*" DE MORGAN'S *Calculus*.

§ 1. In this book we do not propose to attempt to deal thoroughly with the question of convergence of series, partly because a satisfactory treatment would take up too much space and partly because it seems best for a student to study the subject *after* he has had some acquaintance with infinite series; on the other hand we shall try to point out places at which we make assumptions about convergence and at which difficulties arise.

We shall assume that the reader has studied the introduction to the subject of convergence of series in algebra and that, after reading this book, he will study it further in such books as Chrystal's *Algebra*, Hobson's *Trigonometry* and Hardy's *Pure Mathematics*.

Below we give some definitions and state some properties about convergence which we shall assume.

For the proofs of these properties the student should consult the books referred to above.

§ 2. Series of Real Terms.

Def. Consider the infinite series $u_1 + u_2 + u_3 + \dots + u_n + \dots$.

If u_1, u_2, etc. are real, and if $\sum\limits_{r=1}^{r=n} u_r$ tends to a definite limit as n tends to ∞, the series is said to be **convergent** and that limit is called the **sum of the series.**

It should be noted that the word "sum" as applied to an infinite series has an artificial meaning and we must not assume that properties of the sum of a finite series are necessarily true of the sum of an infinite series. All such properties need investigation for the sum of an infinite series; some will be found to be true and some untrue or true only over a limited range.

Def. Let $|u_r|$ denote the positive value of the term u_r (irrespective of the + or − before it); then, if $\Sigma|u_n|$ is convergent the series Σu_n is said to be **absolutely convergent**.

Def. If $\sum_{r=1}^{r=n} u_r$ tends to $\pm\infty$ as n tends to ∞, the series is said to be **divergent**.

§3. Series of Complex Terms.

Def. If $u_1, u_2, \ldots$ are complex and if $u_r = v_r + iw_r$ where v_r and w_r are real, then $u_1 + u_2 + \ldots$ is said to be **convergent** if the two series $v_1 + v_2 + \ldots$ and $w_1 + w_2 + \ldots$ are both convergent.

Def. The definition of an **absolutely convergent** series is the same as the above definition if for "convergent" in each case we read "absolutely convergent."

If $\Sigma \operatorname{mod}(u_n)$ is convergent, then Σu_n is convergent. For a geometrical illustration see § 11.

§4. The Power Series $\Sigma(a_n z^n)$ where a is real.

If the power series is absolutely convergent for any particular value of mod (z), say r, then it is absolutely convergent for all values of mod (z) such that mod $(z) < r$.

Def. If a series is convergent for all values of z such that mod $(z) < r$ and is not convergent for any value of z such that mod $(z) > r$, the circle with centre at the origin of radius r is called the **circle of convergence**.

We may then say that the series is convergent for all points inside the circle of convergence, and is divergent for all points outside it; but the cases of points on the circle of convergence will need separate investigation.

§5. Differentiation and Integration.

If $S(z)$, the sum of a series involving z, is absolutely convergent over any range of values, we shall make the following assumptions.

(i) The result of differentiating the series term by term gives a series whose sum is equal to the result of differentiating $S(z)$, *provided*

that the derived series is absolutely convergent,† and provided z is limited to the range of values mentioned above.

Or, in symbols, if

$$\text{S}(z) = u_1 + u_2 + u_3 + \dots \textit{ ad infin.}$$

then
$$\frac{d\,\text{S}(z)}{dz} = \frac{du_1}{dz} + \frac{du_2}{dz} + \frac{du_3}{dz} + \dots \textit{ ad infin.}$$

(ii) The result of integrating the series term by term gives a series whose sum is equal to the result of integrating $\text{S}(z)$, provided the range of the integration in each case is within the limits mentioned above.

Note that in the process of differentiating a single term we have a quantity which $\to 0$ as we approach the limit. Now in differentiating the separate terms of an infinite series, we have an infinite number of small quantities and, though each $\to 0$, it is by no means certain that their sum will necessarily approach 0.

§6. The Geometrical Series.

The sum of the finite series $1 + z + z^2 + \dots + z^{n-1}$ is

$$\frac{1 - z^n}{1 - z} = \frac{1}{1 - z} - \frac{z^n}{1 - z},$$

whether z be real or complex.

Now suppose $z = x + iy = r(\cos\theta + i\sin\theta)$ or (r, θ).

Then by Demoivre's Theorem, Case 1, since n is a positive integer,

$$z^n = (r^n, n\theta) = r^n(\cos n\theta + i\sin n\theta),$$

$$\therefore \underset{n\to\infty}{\text{Lt}} \frac{z^n}{1 - z} = \underset{n\to\infty}{\text{Lt}} \frac{r^n(\cos n\theta + i\sin n\theta)}{1 - r\cos\theta - ir\sin\theta}$$

$$= 0, \text{ when } r < 1.$$

$\therefore$ Provided $r = \text{mod}\,(z) < 1$,

$1 + z + z^2 + \dots$ *ad infin.* converges and its sum is $\dfrac{1}{1 - z}$

† The following example illustrates that the theorem is not true unless the derived series is convergent.

$$\tfrac{1}{2}x = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \textit{ ad infin.}$$ (See Note, Example iv, p. 387.)

If we differentiate the series, we get

$$\cos x - \cos 2x + \cos 3x - \dots \textit{ ad infin.},$$

which does not converge and is completely indeterminate.

§ 7. The Binomial Series.

$$1+\frac{n}{1}z+\frac{n(n-1)}{1.2}z^2+\frac{n(n-1)(n-2)}{1.2.3}z^3+\dots.$$

When this series converges, its sum is one of the values of $(1+z)^n$ and it is taken to be the *principal value* of $(1+z)^n$.

The reader is already aware that if n is fractional a^n is a many-valued expression.

Case 1. *If n is a positive integer*, we have seen in Chap. XV, § 1, p. 250 that the series is finite for all values of z, and its sum is $(1+z)^n$ which is single-valued in this case.

Case 2. *If n is negative or fractional and* mod (z) *is numerically* < 1, the series is convergent.

Case 3. *If n is negative or fractional and* mod $(z)=1$,

(i) if n is a positive fraction, the series is convergent;

(ii) if n is between 0 and -1, the series is convergent, except in the case of $z=-1$ when it is divergent;

(iii) if $n=-1$ or <-1, the series is divergent.

Case 4. *If n is negative or fractional and* mod $(z)>1$, the series is divergent.

§ 8. The Exponential Series.

$$\exp(z)\equiv 1+\frac{z}{\lfloor 1}+\frac{z^2}{\lfloor 2}+\frac{z^3}{\lfloor 3}+\dots \textit{ ad infin.}$$

This series is absolutely convergent for all values of z real or complex.

In algebra books it is shown that, when z is real, $\exp(z)$ is at any rate one of the values of e^z.

We take $\exp(z)$ as the principal value of e^z.

The series for $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ are absolutely convergent for all values of z, real or complex.

§ 9. The Logarithmic Series.

$$\frac{z}{1}-\frac{z^2}{2}+\frac{z^3}{3}-\dots \textit{ ad infin.}$$

In algebra books it is shown that this series is convergent and its

sum is $\log(1+z)$ provided z is real and lies between ± 1. This is also true when $z = +1$, but not when $z = -1$.

When z is complex, the series is convergent and its sum is $\log(1+z)$ the principal value of $\operatorname{Log}(1+z)$, provided $\bmod(z) < 1$, and also in the case of $\bmod(z) = 1$ except when $\operatorname{amp}(z) = (2n+1)\pi$.

§ 10. Multiplication of Series.

Multiply $u_0 + u_1 + u_2 + u_3 + \ldots$ by $v_0 + v_1 + v_2 + v_3 + \ldots$ and arrange the work as follows:

$$\begin{array}{lllll}
u_0 & +u_1 & +u_2 & +u_3 & +\ldots \\
v_0 & +v_1 & +v_2 & +v_3 & +\ldots \\
\hline
u_0v_0 & +u_1v_0 & +u_2v_0 & +u_3v_0 & +\ldots \\
 & u_0v_1 & +u_1v_1 & +u_2v_1 & +\ldots \\
 & & u_0v_2 & +u_1v_2 & +\ldots \\
 & & & u_0v_3 & +\ldots \\
 & & & & \ldots \\
\hline
w_0 & +w_1 & +w_2 & +w_3 & +\ldots
\end{array}$$

If Σu_n and Σv_n are two series of numbers, real or complex, which are absolutely convergent over a stated range, and if

$$w_n = u_0v_n + u_1v_{n-1} + u_2v_{n-2} + \ldots + u_nv_0,$$

we shall assume (i) that the series Σw_n, which we shall call the product series, is absolutely convergent over the same range and (ii) that the sum of the product series is equal to the products of the sums of the original series over the same range.

§ 11. Geometrical Illustration of Convergence.

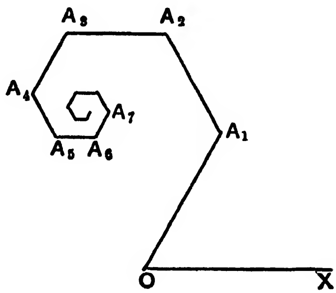
The sum of a series of real positive numbers, $u_1 + u_2 + u_3 + \ldots$, may be represented by measuring off along a straight line OX successive lengths OA_1, A_1A_2, A_2A_3, ... equal to $u_1, u_2, u_3, \ldots$.

The sum of n terms is then represented by OA_n.

If there is a point A towards which A_n continually approaches but which it never reaches such that $\operatorname*{Lt}_{n\to\infty} A_nA = 0$, OA represents the sum to infinity of the series.

Suppose the figure above represents a series of rods hinged at O, A_1, A_2, ...; if we turn the rods so that they make angles θ, 2θ, 3θ, ... respectively with OX, in their new positions OA_1, A_1A_2, A_2A_3, ... represent the successive terms in the following series of complex numbers $(u_1, \theta) + (u_2, 2\theta) + (u_3, 3\theta) + \ldots$ and OA_n represents the sum of the series to n terms.

The figure shows that, in the case taken, the rods form a sort of spiral which coils round and round a point; it suggests that, if we



added on more and more terms, we should get closer and closer to a fixed point. This suggests that A_n approaches a fixed point so that, as n approaches infinity, the sum of the series approaches a fixed value.

The reader is advised to draw out in this way the successive terms of several convergent series of complex numbers.

CHAPTER XIX

SUMMATION OF INFINITE SERIES

§ 1. The most common method of summing an infinite series is to throw the series into the form of some standard series whose sum is known; in summing trigonometrical series this is frequently done by using complex numbers (see Chap. XVI, § 5, p. 271).

In some cases we can sum a series to n terms, and then find the limit of that sum as n "tends to infinity."

In this chapter the convergence of the series is generally assumed, or the limitations under which they are convergent are stated. At a later stage the pupil may investigate the convergence for himself.

§ 2. Series whose summation depends on known series.

Geometrical Series.

For the convergence of these series see Chap. XVIII, § 6, p. 305.

Example i. *Find the sum to infinity of the series* (x *real and* <1)

$$1+2x\cos\theta+2x^2\cos 2\theta+2x^3\cos 3\theta+\ldots.$$

Let $$C=1+2x\cos\theta+2x^2\cos 2\theta+\ldots$$

and $$S=2x\sin\theta+2x^2\sin 2\theta+\ldots.$$

$$\begin{aligned}\therefore\quad C+iS&=1+2(x,\theta)+2(x^2,2\theta)+\ldots\\&=1+2(x,\theta)+2(x,\theta)^2+\ldots\\&=1+\frac{2(x,\theta)}{1-(x,\theta)}\qquad\text{since mod }(x,\theta)=x<1\\&=1+\frac{2(x,\theta)\{1-(x,-\theta)\}}{\{1-(x,\theta)\}\{1-(x,-\theta)\}}\\&=1+\frac{2(x,\theta)-2(x^2,0)}{1-2x\cos\theta+x^2}\\&=\frac{1-x^2+2ix\sin\theta}{1-2x\cos\theta+x^2}.\end{aligned}$$

Hence, equating real parts on each side of the equation, we get

$$C=\frac{1-x^2}{1-2x\cos\theta+x^2}$$

By equating the imaginary parts and dividing by 2, we get

$$x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\ldots ad\ infin.=\frac{x\sin\theta}{1-2x\cos\theta+x^2}.$$

Binomial Series.

For the convergence of these series see Chap. XVIII, § 7, p. 306.

Example ii. *Find the sum to infinity of the series*

$$x \sin\theta + \tfrac{1}{4}x^3 \sin 3\theta + \frac{1.3}{4.8}x^5 \sin 5\theta + \frac{1.3.5}{4.8.12}x^7 \sin 7\theta + \dots$$

for certain real values of x. What are these values?

Let $$S = x\sin\theta + \tfrac{1}{4}x^3 \sin 3\theta + \frac{1.3}{4.8}x^5 \sin 5\theta + \dots$$

and $$C = x\cos\theta + \tfrac{1}{4}x^3 \cos 3\theta + \frac{1.3}{4.8}x^5 \cos 5\theta + \dots.$$

$$\therefore\ C + iS = (x, \theta) + \tfrac{1}{4}(x^3, 3\theta) + \frac{1.3}{4.8}(x^5, 5\theta) + \dots$$

$$= (x, \theta)\left[1 + \tfrac{1}{4}(x^2, 2\theta) + \frac{1.3}{4.8}(x^4, 4\theta) + \dots\right]$$

$$= (x, \theta)\left[1 + \tfrac{1}{2}\left(\frac{x^2}{2}, 2\theta\right) + \frac{\frac{1}{2}.\frac{3}{2}}{\underline{|2}}\left(\frac{x^2}{2}, 2\theta\right)^2 + \dots\right]$$

$$= (x, \theta)\left[1 - \left(\frac{x^2}{2}, 2\theta\right)\right]^{-\frac{1}{2}}$$

provided mod $\left(\frac{x^2}{2}, 2\theta\right)$ is <1, that is $\frac{x^2}{2} < 1$, or $\sqrt{2} > x > -\sqrt{2}$.

$$\therefore\ C + iS = \frac{(x, \theta)}{\left[1 - \frac{x^2}{2}\cos 2\theta - i\frac{x^2}{2}\sin 2\theta\right]^{\frac{1}{2}}}.$$

Let $$r^2 = 1 - x^2\cos 2\theta + \frac{x^4}{4} \quad\text{and}\quad \tan\alpha = \frac{x^2 \sin 2\theta}{2 - x^2\cos 2\theta}.$$

$$\therefore\ C + iS = \frac{(x, \theta)}{(r, -\alpha)^{\frac{1}{2}}} = \frac{(x, \theta)}{\left(\sqrt{r}, -\frac{\alpha}{2}\right)} = \left(\frac{x}{\sqrt{r}}, \theta + \frac{\alpha}{2}\right).$$

Equating the imaginary parts on each side of the equations,

$$S = \frac{x}{\sqrt{r}}\sin\left(\theta + \frac{\alpha}{2}\right).$$

$$\therefore\ S = \frac{x}{\sqrt{1 - x^2\cos 2\theta + \frac{x^4}{4}}}\sin\left(\theta + \frac{\alpha}{2}\right), \text{ where } \tan\alpha = \frac{x^2 \sin 2\theta}{2 - x^2\cos 2\theta}$$

provided $\sqrt{2} > x > -\sqrt{2}$.

Exponential Series.

For the convergence of these series see Chap. XVIII, § 8, p. 306.

Example iii. *Find the sum to infinity of the series*

$$\frac{a^2 \sin 2\theta}{\underline{|2}} + \frac{a^4 \sin 4\theta}{\underline{|4}} + \dots.$$

For what real values of a is the summation justifiable?

Let $$S = \frac{a^2 \sin 2\theta}{\underline{|2}} + \frac{a^4 \sin 4\theta}{\underline{|4}} + \dots$$

and $$C = 1 + \frac{a^2 \cos 2\theta}{\underline{|2}} + \frac{a^4 \cos 4\theta}{\underline{|4}} + \dots.$$

$$\therefore\ C + iS = 1 + \frac{(a^2, 2\theta)}{\underline{|2}} + \frac{(a^4, 4\theta)}{\underline{|4}} + \dots \text{ (see footnote †)}$$

$$= \tfrac{1}{2}\left[1 + \frac{(a, \theta)}{\underline{|1}} + \frac{(a, \theta)^2}{\underline{|2}} + \frac{(a, \theta)^3}{\underline{|3}} + \frac{(a, \theta)^4}{\underline{|4}} + \dots \right.$$
$$\left. + 1 - \frac{(a, \theta)}{\underline{|1}} + \frac{(a, \theta)^2}{\underline{|2}} - \frac{(a, \theta)^3}{\underline{|3}} + \frac{(a, \theta)^4}{\underline{|4}} + \dots\right]$$

$$= \tfrac{1}{2}[\exp(a, \theta) + \exp(-a, \theta)] \quad \text{for all values of } a$$

$$= \tfrac{1}{2}[\exp(a\cos\theta)\exp(ia\sin\theta) + \exp(-a\cos\theta)\exp(-ai\sin\theta)]$$

$$= \tfrac{1}{2}\exp(a\cos\theta)\{\cos(a\sin\theta) + i\sin(a\sin\theta)\}$$
$$+ \tfrac{1}{2}\exp(-a\cos\theta)\{\cos(a\sin\theta) - i\sin(a\sin\theta)\}$$

Equating the imaginary parts on each side,

$$S = \sin(a\sin\theta)\frac{\exp(a\cos\theta) - \exp(-a\cos\theta)}{2}$$

$$= \sin(a\sin\theta)\sinh(a\cos\theta),$$

and the summation is true for all real values of a.

Logarithmic Series.

For the convergence of these series see Chap. XVIII, § 9, p. 306.

Example iv. *Sum to infinity the series*

$$\sin\theta - \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta - \dots,$$

where θ lies between $\pm\frac{\pi}{2}$.

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† Or alternatively from this point

$$C + iS = \cosh(a, \theta) = \cosh(a\cos\theta + ia\sin\theta)$$
$$= \cosh(a\cos\theta)\cos(a\sin\theta) + i\sinh(a\cos\theta)\sin(a\sin\theta).$$
$$\therefore\ S = \sinh(a\cos\theta)\sin(a\sin\theta).$$

This series is most easily summed by aid of Gregory's Series, see Chap. XXI, § 6, p. 334; but the following method of summation is instructive.

Let $$S = \sin\theta - \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta - \ldots,$$

and $$C = \cos\theta - \tfrac{1}{3}\cos 3\theta + \tfrac{1}{5}\cos 5\theta - \ldots.$$

Let $z \equiv (1, \theta)$. Then

$$C + iS = z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots$$

$$= \frac{1}{2i}\left[iz + \frac{(iz)^2}{2} + \frac{(iz)^3}{3} + \ldots \right.$$
$$\left. + iz - \frac{(iz)^2}{2} + \frac{(iz)^3}{3} - \ldots \right]$$

$$= \frac{1}{2i}[\log(1+iz) - \log(1-iz)], \qquad \text{as } \theta \neq (2n+1)\pi$$

$$= -\frac{i}{2}\left[\log\left\{1 + \left(1, \frac{\pi}{2} + \theta\right)\right\} - \log\left\{1 + \left(1, \theta - \frac{\pi}{2}\right)\right\}\right]$$

$$= -\frac{i}{2}\left[\log\left(2\cos\overline{\frac{\pi}{4} + \frac{\theta}{2}}, \frac{\pi}{4} + \frac{\theta}{2}\right) - \log\left(2\cos\overline{\frac{\theta}{2} - \frac{\pi}{4}}, \frac{\theta}{2} - \frac{\pi}{4}\right)\right]$$

$$= -\frac{i}{2}\left[\log\left(2\cos\overline{\frac{\pi}{4} + \frac{\theta}{2}}\right) + i\left(\frac{\pi}{4} + \frac{\theta}{2}\right) - \log\left(2\cos\overline{\frac{\theta}{2} - \frac{\pi}{4}}\right) - i\left(\frac{\theta}{2} - \frac{\pi}{4}\right)\right].$$

Now, as θ lies between $\pm\frac{\pi}{2}$, $\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$ are both positive.

∴ Equating imaginary parts on each side,

$$S = -\tfrac{1}{2}\left[\log\left(2\cos\overline{\frac{\pi}{4} + \frac{\theta}{2}}\right) - \log\left(2\cos\overline{\frac{\theta}{2} - \frac{\pi}{4}}\right)\right]$$

$$= \tfrac{1}{2}\log\frac{\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)}$$

$$= \tfrac{1}{2}\log\frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}$$

$$= \tfrac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

EXERCISE XIX. a.

Find the sum to infinity of the following series:

1. (i) $\sin\theta+\sin^3\theta+\sin^5\theta+\dots,$

(ii) $\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2^2}\cos 3\theta+\dots,$

(iii) $\sin\theta+\frac{1}{3}\sin 3\theta+\frac{1}{3^2}\sin 5\theta+\dots,$

(iv) $x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\dots\quad(x<1),$

(v) $\cos\alpha+x\cos(\alpha+\beta)+x^2\cos(\alpha+2\beta)+\dots\quad(x<1),$

(vi) $1+x\cosh\alpha+x^2\cosh 2\alpha+x^3\cosh 3\alpha+\dots.$ State the necessary conditions.

2. (i) $1+nx\cos\theta+\frac{n(n-1)}{\lfloor 2}x^2\cos 2\theta+\frac{n(n-1)(n-2)}{\lfloor 3}x^3\cos 3\theta+\dots\quad(x<1),$

(ii) $\frac{1}{2}\sin\theta+\frac{1.3}{2.4}\sin 2\theta+\frac{1.3.5}{2.4.6}\sin 3\theta+\dots\quad(2\pi>\theta>0),$

(iii) $n\cos\theta+\frac{n(n+1)}{1.2}\cos 2\theta+\frac{n(n+1)(n+2)}{1.2.3}\cos 3\theta+\dots$

$(n<1 \text{ and } 2\pi>\theta>0),$

(iv) $\frac{1}{2}\sin\theta-\frac{1}{2.4}\sin 2\theta+\frac{1.3}{2.4.6}\sin 3\theta-\dots\quad(\theta \text{ between } \pm\pi),$

(v) $1+\frac{1}{3}x\cos\theta+\frac{1.4}{3.6}x^2\cos 2\theta+\frac{1.4.7}{3.6.9}x^3\cos 3\theta+\dots\quad(x<1).$

3. (i) $\frac{\sin x}{\lfloor 1}-\frac{\sin 2x}{\lfloor 2}+\frac{\sin 3x}{\lfloor 3}-\dots,$

(ii) $x\sin\alpha+\frac{x^2}{\lfloor 2}\sin 2\alpha+\frac{x^3}{\lfloor 3}\sin 3\alpha+\dots,$

(iii) $1-x\cos\theta+\frac{x^2}{\lfloor 2}\cos 2\theta-\dots,$

(iv) $\cos\theta+\cos(\theta+\alpha)+\frac{1}{\lfloor 2}\cos(\theta+2\alpha)+\dots,$

(v) $\cos\alpha+\frac{1}{\lfloor 2}\cos(\alpha+\beta)+\frac{1}{\lfloor 4}\cos(\alpha+2\beta)+\dots.$

4. (i) $\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\dots\quad(x \text{ between } \pm\pi),$

(ii) $\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{3}\cos 3\theta+\dots\quad(2\pi>\theta>0),$

(iii) $a\sin x-\frac{a^2}{2}\sin 2x+\frac{a^3}{3}\sin 3x-\dots\quad(a<1),$

(iv) $\cos x+\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x+\dots$

(*a*) when $\pi>x>0$, (*b*) when $2\pi>x>\pi$.

5. $x\sinh\alpha + x^2\sinh 2\alpha + x^3\sinh 3\alpha + \ldots$. State the necessary conditions.

6. $\cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \cos^3\theta\cos 3\theta + \ldots$ $(\theta \neq n\pi)$.

7. $\cos\theta\sin 2\theta + \cos^2\theta\sin 3\theta + \cos^3\theta\sin 4\theta + \ldots$ $(\theta \neq n\pi)$.

8. $\sin\theta\sin\theta + \sin^2\theta\sin 2\theta + \sin^3\theta\sin 3\theta + \ldots$ $\left(\theta \neq n\pi \pm \frac{\pi}{2}\right)$.

9. $\cos\alpha + \dfrac{\cos\beta}{\underline{|1}}\cos(\alpha+\beta) + \dfrac{\cos^2\beta}{\underline{|2}}\cos(\alpha+2\beta) + \ldots$.

10. $\sin\theta + nx\sin 3\theta + \dfrac{n(n+1)}{1.2}x^2\sin 5\theta + \ldots$ $(x<1)$.

11. $\dfrac{\sin x}{\pi} - \dfrac{\sin 2x}{\pi^2} + \dfrac{\sin 3x}{\pi^3} - \ldots$.

12. $x\sin\alpha - \dfrac{x^2}{2}\sin(\alpha+\beta) + \dfrac{x^3}{3}\sin(\alpha+2\beta) - \ldots$ $(x<1)$.

13. $1 + x\sin\alpha\cos\alpha + \dfrac{x^2}{\underline{|2}}\sin^2\alpha\cos 2\alpha + \dfrac{x^3}{\underline{|3}}\sin^3\alpha\cos 3\alpha + \ldots$.

14. $1 + \cos\theta\tan\theta + \dfrac{1}{\underline{|2}}\cos 2\theta\tan^2\theta + \dfrac{1}{\underline{|3}}\cos 3\theta\tan^3\theta + \ldots$.

15. $\cos\alpha + \dfrac{1}{\underline{|3}}\cos(\alpha+2\beta) + \dfrac{1}{\underline{|5}}\cos(\alpha+4\beta) + \ldots$.

16. $\cos\theta\sin\theta + \frac{1}{2}\cos^2\theta\sin 2\theta + \frac{1}{3}\cos^3\theta\sin 3\theta + \ldots$ (θ acute).

17. $\cos\theta\sec\theta + \dfrac{1}{2}\cos 2\theta\sec^2\theta + \dfrac{1}{2^2}\cos 3\theta\sec^3\theta + \ldots$ $(\sec\theta < 2)$.

18. $\cos\alpha + nx\cos(\alpha+\theta) + \dfrac{n(n-1)}{1.2}x^2\cos(\alpha+2\theta) + \ldots$ $(x<1)$.

19. $\cos\alpha\cos\beta + \frac{1}{2}\cos 2\alpha\cos 2\beta + \frac{1}{3}\cos 3\alpha\cos 3\beta + \ldots$ $(\alpha \pm \beta \neq 2n\pi)$.

20. $\cos^2\theta - \dfrac{1}{2}\cos^2 2\theta + \dfrac{1}{2^2}\cos^2 3\theta - \ldots$.

21. Prove that the value of the series

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1.3}{2.4}\cos 3\theta + \frac{1.3.5}{2.4.6}\cos 4\theta + \ldots,$$

is one of the values of $\cos\left(\dfrac{\pi}{4} + \dfrac{3\theta}{4}\right)\left(2\sin\dfrac{\theta}{2}\right)^{-\frac{1}{2}}$ if θ lies between certain limits; and find those limits.

For further examples, the student may sum to infinity the series given in Exercise XVI *d*, p. 272, Nos. 18 $(x<1)$, 19 $(x<1)$, 20, 29 and Miscellaneous Exercises III, pp. 286–287, Nos. 106 $(x<1)$, 113 $(x<1)$.

§ 3. Many infinite series may be summed by the method of differences. The sum to n terms is first found and the limit of the sum as n tends to infinity. See Chap. XVI, § 3, p. 267.

§ 4. The sum of a series can sometimes be found by differentiating or integrating a known series and its sum. For the conditions under which this is justifiable see Chap. XVIII, § 5, p. 304. Also see Chap. XXI, § 5, Example ii, p. 332.

EXERCISE XIX. b.

Find the sum to infinity of the following series:

Nos. 1–5. Use the method of differences.

1. $\sin\alpha\sin 3\alpha+\sin\frac{\alpha}{2}\sin\frac{3\alpha}{2}+\sin\frac{\alpha}{2^2}\sin\frac{3\alpha}{2^2}+\ldots.$

2. $\sin^2\frac{\alpha}{2}\operatorname{cosec}2\alpha+\frac{1}{2}\sin^2\frac{\alpha}{2^2}\operatorname{cosec}\alpha+\frac{1}{2^2}\sin^2\frac{\alpha}{2^3}\operatorname{cosec}\frac{\alpha}{2}+\ldots.$

3. $\sin^3\frac{\alpha}{3}+3\sin^3\frac{\alpha}{3^2}+3^2\sin^3\frac{\alpha}{3^3}+\ldots.$

4. $\sin 2\theta\cos^2\theta-\frac{1}{2}\sin 4\theta\cos^2 2\theta+\frac{1}{4}\sin 8\theta\cos^2 4\theta-\ldots.$

5. $\tan^{-1}\frac{2}{4}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{2}{16}+\ldots.$

Nos. 6–8. See Chap. XVI, Example iv, p. 265.

6. $\cos\theta+x\cos 2\theta+x^2\cos 3\theta+\ldots\quad(x<1).$

7. $\cos\alpha+x\cos(\alpha+\beta)+x^2\cos(\alpha+2\beta)+\ldots\quad(x<1).$

8. $\sin\alpha+x\sin(\alpha+\beta)+x^2\sin(\alpha+2\beta)+\ldots\quad(x<1).$

Nos. 9–13. Use differentiation or integration.

9. Sum $\cos\theta+2\cos 2\theta+3\cos 3\theta+\ldots$ to n terms.

10. Sum $\sec^2 x+\frac{1}{4}\sec^2\frac{x}{2}+\frac{1}{4^2}\sec^2\frac{x}{2^2}+\ldots$ *ad infin.*

11. Show that $\sin\theta+\frac{1}{2}\frac{\sin^3\theta}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{\sin^5\theta}{5}+\ldots$ *ad infin.* $=\theta$, provided θ lies between $\pm\frac{\pi}{2}$.

12. Show that, provided $2\pi>x>0$,

$$\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots \textit{ ad infin.}=\frac{x^2}{4}-\frac{\pi x}{2}+S,$$

where

$$S=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots \textit{ ad infin.}$$

18. Show that, provided x lies between $\pm\pi$,

$$\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \textit{ ad infin.} = S - \frac{x^2}{4},$$

where
$$S = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \textit{ ad infin.}$$

*EXERCISE XIX. c.

MISCELLANEOUS SERIES.

1. Prove that, if $\theta \neq 2n\pi$, $\log \cot \theta = \cos 2\theta + \frac{1}{3}\cos^3 2\theta + \frac{1}{5}\cos^5 2\theta + \dots$ *ad infin.*

2. Prove that $\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{4}\tan\frac{\theta}{4} + \dots$ *ad infin.* $= \frac{1}{\theta} - 2\cot 2\theta$.

3. Prove that $\tan\frac{x}{2}\sec x + \tan\frac{x}{2^2}\sec\frac{x}{2} + \tan\frac{x}{2^3}\sec\frac{x}{2^2} + \dots$ *ad infin.* $= \tan x$.

4. Prove that, if $\cos\alpha < \cos(\alpha - \beta)$,

$$1 + \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} + \frac{\cos\alpha\cos(\alpha+2\beta)}{\cos^2(\alpha-\beta)} + \frac{\cos^2\alpha\cos(\alpha+3\beta)}{\cos^3(\alpha-\beta)} + \dots \textit{ ad infin.} = 0.$$

5. Sum to infinity $a\sin\theta + \frac{1}{3}a^3\sin 3\theta + \frac{1}{5}a^5\sin 5\theta + \dots$, where $0 < a < 1$. Deduce the sum of the series $\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \dots$

(i) when $0 < \theta < \pi$, (ii) when $\pi < \theta < 2\pi$.

6. Prove that $\frac{\pi}{4}$ is the sum of the infinite series

$$\tan^{-1}\tfrac{1}{3} + \tan^{-1}\tfrac{1}{7} + \dots + \tan^{-1}\frac{1}{n^2+n+1} + \dots.$$

7. Prove that $\cot^{-1}(2 \,.\, 1^2) + \cot^{-1}(2 \,.\, 2^2) + \cot^{-1}(2 \,.\, 3^2) + \dots$ *ad infin.* $= \frac{\pi}{4}$.

8. If θ lies between $\frac{\pi}{2}$ and π, prove that

$$\cot\frac{\theta}{2} + \tfrac{1}{3}\cot^3\frac{\theta}{2} + \tfrac{1}{5}\cot^5\frac{\theta}{2} + \dots \textit{ ad infin.}$$
$$= \tfrac{1}{2}\left[\sin\theta + \tfrac{1}{3}\sin^3\theta + \tfrac{1}{5}\sin^5\theta + \dots \textit{ ad infin.}\right],$$

and, if θ lies between 0 and $\frac{\pi}{2}$,

$$\tan^2\frac{\theta}{2} + \tfrac{1}{3}\tan^6\frac{\theta}{2} + \tfrac{1}{5}\tan^{10}\frac{\theta}{2} + \dots \textit{ ad infin.}$$
$$= \tfrac{1}{2}\left[\tfrac{1}{2}\sin^2\theta + \tfrac{1}{4}\sin^4\theta + \tfrac{1}{6}\sin^6\theta + \dots \textit{ ad infin.}\right].$$

9. Prove that the sum of the infinite series

$$1+\tfrac{1}{2}\cos^2\theta+\frac{1.3}{2.4}\cos^2\theta\cos 2\theta+\frac{1.3.5}{2.4.6}\cos^3\theta\cos 3\theta+\ldots$$

is
$$\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\Big/\sqrt{\sin\theta} \text{ when } 0<\theta<\pi,$$

and
$$\cos\left(\frac{\theta}{2}-\frac{3\pi}{4}\right)\Big/\sqrt{-\sin\theta} \text{ when } \pi<\theta<2\pi.$$

10. Find the sum to infinity of

$$\sin x+\cos y\sin(x+y)+\frac{\cos^2 y}{1.2}\sin(x+2y)+\ldots.$$

11. Prove that, if x lies between 0 and 2π,

$$\frac{\sin 2x}{1.3}+\frac{\sin 3x}{2.4}+\frac{\sin 4x}{3.5}+\ldots \textit{ ad infin.}=\tfrac{1}{4}\sin x\left[1-4\log\left(2\sin\frac{x}{2}\right)\right].$$

12. Find the sum to infinity of

$$\cos^2\alpha-\tfrac{1}{2}\sin^2 2\alpha+\tfrac{1}{3}\cos^2 3\alpha-\tfrac{1}{4}\sin^2 4\alpha+\ldots.$$

13. Find the sum to infinity of

$$\sin x-n\cos y\sin(x+y)+\frac{n(n-1)}{\lfloor 2}\cos^2 y\sin(x+2y)-\ldots.$$

14. Sum $\sin(n+r)\theta\,\dfrac{\sin n\theta}{\sin r\theta}-\tfrac{1}{2}\sin(n+r)2\theta\left(\dfrac{\sin n\theta}{\sin r\theta}\right)^2$

$$+\tfrac{1}{3}\sin(n+r)3\theta\left(\frac{\sin n\theta}{\sin r\theta}\right)^3-\ldots \textit{ ad infin.}$$

15. Sum $\cos\theta\sin\theta+\dfrac{\cos^3\theta\sin 3\theta}{\lfloor 3}+\dfrac{\cos^5\theta\sin 5\theta}{\lfloor 5}+\ldots$ *ad infin.*

16. Prove that

$$\cos\theta\frac{\cos\theta}{1}-\sin 2\theta\frac{\cos^2\theta}{2}-\cos 3\theta\frac{\cos^3\theta}{3}+\sin 4\theta\frac{\cos^4\theta}{4}+\cos 5\theta\frac{\cos^5\theta}{5}-\ldots \textit{ ad infin.}$$
$$=\cot^{-1}(1+\tan\theta+\tan^2\theta).$$

17. Find the sums to infinity of the following series (θ between 0 and 2π):

(i) $\dfrac{\cos\theta}{1.2}+\dfrac{\cos 2\theta}{2.3}+\dfrac{\cos 3\theta}{3.4}+\ldots,$

(ii) $\dfrac{\sin\theta}{1.2}+\dfrac{\sin 2\theta}{2.3}+\dfrac{\sin 3\theta}{3.4}+\ldots.$

18. Prove that

$$5\cos\theta+\frac{7}{\lfloor 3}\cos 3\theta+\frac{9}{\lfloor 5}\cos 5\theta+\ldots \textit{ ad infin.}$$

$$=\tfrac{1}{2}e^{\cos\theta}\cos(\theta+\sin\theta)+\tfrac{1}{2}e^{-\cos\theta}\cos(\theta-\sin\theta)+4\cos(\sin\theta)\sinh(\cos\theta).$$

19. Prove that

$$x+\frac{x^4}{\underline{|4}}+\frac{x^7}{\underline{|7}}+\dots\ ad\ infin.=\tfrac{1}{3}e^x+\tfrac{2}{3}e^{-\frac{1}{2}x}\sin\left(x\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right).$$

20. Prove that, when m is a positive integer,

$$\cos m\theta\operatorname{cosec}^m\theta=1+m\operatorname{cosec}\phi\cos\phi+\frac{m(m-1)}{1.2}\operatorname{cosec}^2\phi\cos 2\phi$$
$$+\frac{m(m-1)(m-2)}{1.2.3}\operatorname{cosec}^3\phi\cos 3\phi+\dots,$$

where $\cot\theta=1+\cot\phi$.

21. Show that

$$1+\frac{\cos 4\theta}{\underline{|4}}+\frac{\cos 8\theta}{\underline{|8}}+\frac{\cos 12\theta}{\underline{|12}}+\dots\ ad\ infin.$$
$$=\tfrac{1}{2}[\cos(\cos\theta)\cosh(\sin\theta)+\cosh(\cos\theta)\cos(\sin\theta)].$$

22. Sum the series $\dfrac{\cos 2\phi}{\underline{|2}}+\dfrac{\cos 6\phi}{\underline{|6}}+\dfrac{\cos 10\phi}{\underline{|10}}+\dots\ ad\ infin.$

23. Prove that, if $p<1$, the infinite series

$$1+np\cos\theta+\frac{n(n+1)}{\underline{|2}}p^2\cos 2\theta+\frac{n(n+1)(n+2)}{\underline{|3}}p^3\cos 3\theta+\dots=q^n\cos n\phi,$$

where $\qquad \dfrac{1}{q^2}=1-2p\cos\theta+p^2$ and $\sin\phi=pq\sin\theta$.

24. Sum the series

$$\frac{\cos^3\alpha\cos 3\alpha}{\underline{|3}}+\frac{\cos^7\alpha\cos 7\alpha}{\underline{|7}}+\frac{\cos^{11}\alpha\cos 11\alpha}{\underline{|11}}+\dots\ ad\ infin.$$

CHAPTER XX

EXPANSIONS IN FINITE SERIES

§ 1. In this chapter we expand $\cos n\theta$ and $\dfrac{\sin n\theta}{\sin\theta}$ in series of powers of $\cos\theta$ or $\sin\theta$ and $\cos^n\theta$ and $\sin^n\theta$ in series of cosines and sines of multiples of θ.

§ 2. To express $\cos n\theta$, where n is a positive integer, in a series of descending powers of $\cos\theta$.

In Chap. XIV, § 16, p. 244, it was shown that it is possible to express $\cos n\theta$ in a series of powers of $\cos\theta$ and that the highest term is $2^{n-1}\cos^n\theta$. The whole series is best obtained as follows.

The following expansion was obtained in Chap. XIX, § 2, p. 309, or it can be verified by actual multiplication.

When $x<1$,

$$\frac{1-x^2}{1-2x\cos\theta+x^2}=1+2x\cos\theta+2x^2\cos 2\theta+2x^3\cos 3\theta+\dots\quad\text{(i)}$$

Now
$$\frac{1-x^2}{1-2x\cos\theta+x^2}=(1-x^2)\{1-x(2\cos\theta-x)\}^{-1}$$
$$=(1-x^2)\{1+x(2\cos\theta-x)+x^2(2\cos\theta-x)^2+\dots$$
$$+x^{n-1}(2\cos\theta-x)^{n-1}+x^n(2\cos\theta-x)^n+\dots\},\quad\text{(ii)}$$

since x can be chosen so that $x(2\cos\theta-x)<1$.

Equating coefficients of x^n in (i) and (ii),

$$2\cos n\theta=(2\cos\theta)^n-{}_{n-1}C_1(2\cos\theta)^{n-2}+{}_{n-2}C_2(2\cos\theta)^{n-4}-\dots$$
$$+(-1)^r\,{}_{n-r}C_r(2\cos\theta)^{n-2r}+\dots$$
$$-\{(2\cos\theta)^{n-2}-{}_{n-3}C_1(2\cos\theta)^{n-4}+\dots$$
$$+(-1)^r\,{}_{n-r-1}C_{r-1}(2\cos\theta)^{n-2r}+\dots\}.\quad\text{(iii)}$$

†The coefficient of $(2\cos\theta)^{n-2r}$ in (iii) (when $r>1$)

$$= (-1)^r \{{}_{n-r}C_r + {}_{n-r-1}C_{r-1}\}$$

$$= (-1)^r \left\{ \frac{\lfloor n-r}{\lfloor r \lfloor n-2r} + \frac{\lfloor n-r-1}{\lfloor r-1 \lfloor n-2r} \right\}$$

$$= (-1)^r \frac{\lfloor n-r-1}{\lfloor r \lfloor n-2r} (n-r+r)$$

$$= (-1)^r \frac{n \lfloor n-r-1}{\lfloor r \lfloor n-2r}$$

$$= (-1)^r \frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{\lfloor r},$$

$$\therefore \quad 2\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{\lfloor 2}(2\cos\theta)^{n-4}$$
$$- \frac{n(n-4)(n-5)}{\lfloor 3}(2\cos\theta)^{n-6} + \dots$$
$$+ (-1)^r \frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{\lfloor r}(2\cos\theta)^{n-2r} + \dots. \quad (1)$$

When n is odd, the last term is $(-1)^{\frac{n-1}{2}} n(2\cos\theta)$; when n is even, it is $(-1)^{\frac{n}{2}} 2$.

§ 3. To express $\dfrac{\sin n\theta}{\sin\theta}$, where n is a positive integer, in a series of descending powers of cos θ.

In Chap. xiv, § 16, p. 245, it was shown that it is possible to express $\dfrac{\sin n\theta}{\sin\theta}$ in a series of powers of $\cos\theta$ and that the highest term is $2^{n-1}\cos^{n-1}\theta$. The whole series is best obtained as follows.

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† Before dealing with the general term, the reader should pick out the first few terms of the series for himself.

The following expansion was obtained in Chap. XIX, § 2, p. 309, or it can be verified by actual multiplication.

When $x < 1$,

$$\frac{x \sin\theta}{1 - 2x\cos\theta + x^2} = x\sin\theta + x^2\sin 2\theta + x^3 \sin 3\theta + \dots \quad \text{(i)}$$

Now
$$\frac{x \sin\theta}{1 - 2x\cos\theta + x^2} = x\sin\theta\,\{1 - x(2\cos\theta - x)\}^{-1}$$
$$= x\sin\theta\,\{1 + x(2\cos\theta - x) + x^2(2\cos\theta - x)^2 + \dots$$
$$+ x^{n-1}(2\cos\theta - x)^{n-1} + \dots\}. \quad \text{(ii)}$$

Equating coefficients of x^n in (i) and (ii),

$$\sin n\theta = \sin\theta\,\{(2\cos\theta)^{n-1} - {}_{n-2}C_1(2\cos\theta)^{n-3} + {}_{n-3}C_2(2\cos\theta)^{n-5} - \dots$$
$$+ (-1)^r\,{}_{n-r-1}C_r(2\cos\theta)^{n-2r-1} + \dots\},$$

$$\therefore\ \frac{\sin n\theta}{\sin\theta} = (2\cos\theta)^{n-1} - {}_{n-2}C_1(2\cos\theta)^{n-3} + {}_{n-3}C_2(2\cos\theta)^{n-5} - \dots$$
$$+ (-1)^r\,{}_{n-r-1}C_r(2\cos\theta)^{n-2r-1} + \dots. \quad (2)$$

When n is odd, the last term is $(-1)^{\frac{n-1}{2}}$; when n is even, it is $(-1)^{\frac{n}{2}-1} n\cos\theta$.

§ 4. To express cos nθ, where n is a positive integer, in a series of ascending powers of cos θ. †

Case 1. *When n is even.*

In § 2, it was seen that the general term ‡ in the expansion of $2\cos n\theta$ is

$$(-1)^r \frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{\lfloor r}(2\cos\theta)^{n-2r},$$

where r ranges from 0 to $\frac{n}{2}$.

Let $r = \frac{n}{2} - p$, where p ranges from $\frac{n}{2}$ to 0.

† For an alternative method, see Ex. XX. a, No. 5, p. 326.

‡ Before dealing with the general term, the reader should pick out the first few terms for himself.

The general term

$$=(-1)^r \frac{n\,\lfloor n-r-1}{\lfloor r\,\lfloor n-2r}(2\cos\theta)^{n-2r}$$

$$=(-1)^{\frac{n}{2}-p}\frac{n\,\lfloor n/2+p-1}{\lfloor n/2-p\,\lfloor 2p}(2\cos\theta)^{2p}$$

$$=(-1)^{\frac{n}{2}-p}\,n\,\frac{\left\{\begin{array}{c}\left(\frac{n}{2}+p-1\right)\left(\frac{n}{2}+p-2\right)\ldots\left(\frac{n}{2}+1\right)\\ \frac{n}{2}\left(\frac{n}{2}-1\right)\ldots\left(\frac{n}{2}-p+2\right)\left(\frac{n}{2}-p+1\right)\end{array}\right\}}{\lfloor 2p}(2\cos\theta)^{2p}$$

$$=(-1)^{\frac{n}{2}-p}\,2n\,\frac{\left\{\begin{array}{c}(n+\overline{2p-2})(n+\overline{2p-4})\ldots(n+2)\,n\\ (n-2)\ldots(n-\overline{2p-4})(n-\overline{2p-2})\end{array}\right\}}{\lfloor 2p}\cos^{2p}\theta$$

$$=(-1)^{\frac{n}{2}-p}\,2\,\frac{n^2(n^2-2^2)(n^2-4^2)\ldots(n^2-\overline{2p-2}^{\,2})}{\lfloor 2p}\cos^{2p}\theta;$$

therefore, when n is even,

$$\cos n\theta=(-1)^{\frac{n}{2}}\left\{1-\frac{n^2}{\lfloor 2}\cos^2\theta+\frac{n^2(n^2-2^2)}{\lfloor 4}\cos^4\theta-\ldots\right.$$

$$+(-1)^p\frac{n^2(n^2-2^2)(n^2-4^2)\ldots(n^2-\overline{2p-2}^{\,2})}{\lfloor 2p}\cos^{2p}\theta+\ldots$$

$$\left.+(-1)^{\frac{n}{2}}2^{n-1}\cos^n\theta\right\}. \quad (3)$$

Case 2. When n is odd.

The general term in the expansion of $2\cos n\theta$ is

$$(-1)^r\frac{n\,\lfloor n-r-1}{\lfloor r\,\lfloor n-2r}(2\cos\theta)^{n-2r},$$

where r ranges from 0 to $\frac{n-1}{2}$.

Let $r=\frac{n-1}{2}-p$, where p ranges from $\frac{n-1}{2}$ to 0.

The general term $=(-1)^{\frac{n-1}{2}+p}\,\dfrac{n\,\underline{|(n-1)/2+p}}{\underline{|(n-1)/2-p}\,\underline{|2p+1}}\,(2\cos\theta)^{2p+1}$

$$=(-1)^{\frac{n-1}{2}+p}\,n\,\frac{\left\{\dfrac{\left(\frac{n-1}{2}+p\right)\left(\frac{n-1}{2}+p-1\right)\cdots\left(\frac{n-1}{2}+1\right)}{\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}-1\right)\cdots\left(\frac{n-1}{2}-p+1\right)}\right\}}{\underline{|2p+1}}\,(2\cos\theta)^{2p+1}$$

$$=(-1)^{\frac{n-1}{2}+p}\,2n\,\frac{\left\{\dfrac{(n+\overline{2p-1})\,(n+\overline{2p-3})\,\ldots\,(n+1)}{(n-1)\ldots(n-\overline{2p-3})\,(n-\overline{2p-1})}\right\}}{\underline{|2p+1}}\,\cos^{2p+1}\theta$$

$$=(-1)^{\frac{n-1}{2}+p}\,2n\,\frac{(n^2-1^2)\,(n^2-3^2)\,\ldots\,(n^2-\overline{2p-1}^2)}{\underline{|2p+1}}\,\cos^{2p+1}\theta;$$

therefore, when n is odd,

$$\cos n\theta=(-1)^{\frac{n-1}{2}}\left\{\frac{n}{\underline{|1}}\cos\theta-\frac{n\,(n^2-1^2)}{\underline{|3}}\cos^3\theta\right.$$

$$+\frac{n\,(n^2-1^2)\,(n^2-3^2)}{\underline{|5}}\cos^5\theta-\ldots$$

$$+(-1)^p\,\frac{n\,(n^2-1^2)\,(n^2-3^2)\,\ldots\,(n^2-\overline{2p-1}^2)}{\underline{|2p+1}}\cos^{2p+1}\theta+\ldots$$

$$\left.+(-1)^{\frac{n-1}{2}}\,2^{n-1}\cos^n\theta\right\}. \qquad (4)$$

§ 5. To express $\dfrac{\sin n\theta}{\sin\theta}$, where n is a positive integer, in a series of ascending powers of $\cos\theta$.

Case 1. *When n is even.*

In § 3, it was seen that the general term in the expansion of $\dfrac{\sin n\theta}{\sin\theta}$ is

$$(-1)^r\,{}_{n-r-1}C_r\,(2\cos\theta)^{n-2r-1}=(-1)^r\,\frac{\underline{|n-r-1}}{\underline{|r}\,\underline{|n-2r-1}}\,(2\cos\theta)^{n-2r-1},$$

where r ranges from 0 to $\dfrac{n}{2}-1$.

Let $r = \frac{n}{2} - p - 1$, where p ranges from $\frac{n}{2} - 1$ to 0.

$$\text{The general term} = (-1)^{\frac{n}{2}-p-1} \frac{\lfloor n/2 + p}{\lfloor n/2 - p - 1 \, \lfloor 2p+1} (2\cos\theta)^{2p+1}$$

$$= (-1)^{\frac{n}{2}-p-1} \frac{\left\{ \dfrac{\left(\frac{n}{2}+p\right)\left(\frac{n}{2}+p-1\right)\cdots\left(\frac{n}{2}+1\right)}{\frac{n}{2}\left(\frac{n}{2}-1\right)\cdots\left(\frac{n}{2}-p+1\right)\left(\frac{n}{2}-p\right)} \right\}}{\lfloor 2p+1} (2\cos\theta)^{2p+1}$$

$$= (-1)^{\frac{n}{2}-p-1} \frac{\left\{ \dfrac{(n+2p)(n+\overline{2p-2})\ldots(n+2)}{n(n-2)\ldots(n-\overline{2p-2})(n-2p)} \right\}}{\lfloor 2p+1} \cos^{2p+1}\theta$$

$$= (-1)^{\frac{n}{2}-p-1} \frac{n(n^2-2^2)(n^2-4^2)\ldots(n^2-\overline{2p}^2)}{\lfloor 2p+1} \cos^{2p+1}\theta;$$

therefore, when n is even,

$$\frac{\sin n\theta}{\sin\theta} = (-1)^{\frac{n}{2}-1} \left\{ n\cos\theta - \frac{n(n^2-2^2)}{\lfloor 3} \cos^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{\lfloor 5} \cos^5\theta - \ldots \right.$$

$$+ (-1)^p \frac{n(n^2-2^2)(n^2-4^2)\ldots(n^2-\overline{2p}^2)}{\lfloor 2p+1} \cos^{2p+1}\theta + \ldots$$

$$\left. + (-1)^{\frac{n}{2}+1} 2^{n-1}\cos^{n-1}\theta \right\}. \qquad (5)$$

Case 2. *When n is odd.*

The general term in the expansion of $\frac{\sin n\theta}{\sin\theta}$ is

$$(-1)^r \frac{\lfloor n-r-1}{\lfloor r \, \lfloor n-2r-1} (2\cos\theta)^{n-2r-1},$$

where r ranges from 0 to $\frac{n-1}{2}$.

Let $r = \frac{n-1}{2} - p$, where p ranges from $\frac{n-1}{2}$ to 0.

The general term $= (-1)^{\frac{n-1}{2}-p} \dfrac{\lfloor (n-1)/2+p}{\lfloor (n-1)/2-p \lfloor 2p} (2\cos\theta)^{2p}$

$$= (-1)^{\frac{n-1}{2}-p} \frac{\left\{ \begin{array}{l} \left(\frac{n-1}{2}+p\right)\left(\frac{n-1}{2}+p-1\right)\dots\left(\frac{n-1}{2}+1\right) \\ \left(\frac{n-1}{2}\right)\dots\left(\frac{n-1}{2}-p+2\right)\left(\frac{n-1}{2}-p+1\right)\end{array}\right\}}{\lfloor 2p} (2\cos\theta)^{2p}$$

$$= (-1)^{\frac{n-1}{2}-p} \frac{\left\{ \begin{array}{l} (n+\overline{2p-1})(n+\overline{2p-3})\dots(n+1) \\ (n-1)\dots(n-\overline{2p-3})(n-\overline{2p-1})\end{array}\right\}}{\lfloor 2p} \cos^{2p}\theta$$

$$= (-1)^{\frac{n-1}{2}-p} \frac{(n^2-1^2)(n^2-3^2)\dots(n^2-\overline{2p-1}^2)}{\lfloor 2p} \cos^{2p}\theta;$$

therefore, when n is odd,

$$\frac{\sin n\theta}{\sin\theta} = (-1)^{\frac{n-1}{2}} \left\{ 1 - \frac{n^2-1^2}{\lfloor 2}\cos^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{\lfloor 4}\cos^4\theta - \dots \right.$$

$$+ (-1)^p \frac{(n^2-1^2)(n^2-3^2)\dots(n^2-\overline{2p-1}^2)}{\lfloor 2p}\cos^{2p}\theta + \dots$$

$$\left. + (-1)^{\frac{n-1}{2}} 2^{n-1}\cos^{n-1}\theta \right\}. \quad (6)$$

§ 6. If $\frac{\pi}{2}-\theta$ is substituted for θ in formulae (3), (4), (5), (6) of §§ 4, 5, we obtain:

From (3), when n is even,

$$\cos n\theta = 1 - \frac{n^2}{\lfloor 2}\sin^2\theta + \frac{n^2(n^2-2^2)}{\lfloor 4}\sin^4\theta - \dots + (-1)^{\frac{n}{2}} 2^{n-1}\sin^n\theta. \quad (7)$$

From (4), when n is odd,

$$\sin n\theta = \frac{n}{\lfloor 1}\sin\theta - \frac{n(n^2-1^2)}{\lfloor 3}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{\lfloor 5}\sin^5\theta - \dots$$

$$+ (-1)^{\frac{n-1}{2}} 2^{n-1}\sin^n\theta. \quad (8)$$

From (5), when n is even,

$$\frac{\sin n\theta}{\cos\theta} = n\sin\theta - \frac{n(n^2-2^2)}{\underline{|3}}\sin^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{\underline{|5}}\sin^5\theta - \dots$$

$$+ (-1)^{\frac{n}{2}+1}\, 2^{n-1}\sin^{n-1}\theta. \quad (9)$$

From (6), when n is odd,

$$\frac{\cos n\theta}{\cos\theta} = 1 - \frac{n^2-1^2}{\underline{|2}}\sin^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{\underline{|4}}\sin^4\theta - \dots$$

$$+ (-1)^{\frac{n-1}{2}}\, 2^{n-1}\sin^{n-1}\theta. \quad (10)$$

§7. Series of cosh nu and $\frac{\sinh nu}{\sinh u}$.

Corresponding to the first equations of §§ 2, 3, we have (see Ex. XIX. a, No. 5), provided x is small enough,

$$\frac{1-x^2}{1-2x\cosh\theta + x^2} = 1 + 2x\cosh\theta + 2x^2\cosh 2\theta + \dots$$

and
$$\frac{x\sinh\theta}{1-2x\cosh\theta + x^2} = x\sinh\theta + x^2\sinh 2\theta + \dots,$$

therefore formulae (1) to (6) are equally true if cosh and sinh are substituted throughout for cos and sin respectively.

NOTE. The above results are readily obtained if we assume that formulae (1) to (6) are true for imaginary angles—a big assumption—and if we substitute $i\theta$ for θ and use the facts that $\cos i\theta = \cosh\theta$ and $\sin i\theta = i\sinh\theta$.

If we make the same substitution in formulae (7) to (10), we find that the signs before the terms are now all positive.

EXERCISE XX. a.

The following examples should be worked out by the methods used in §§ 2–6; not by substituting in the results there obtained.

Easier examples will be found in Ex. XIV. d, p. 245.

1. $\frac{\sin 8\theta}{\sin\theta} = 128\cos^7\theta - 192\cos^5\theta + 80\cos^3\theta - 8\cos\theta.$

2. $\cos 8\theta = 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta.$

3. $\frac{\sin 9\theta}{\sin\theta} = 256\cos^8\theta - 448\cos^6\theta + 240\cos^4\theta - 40\cos^2\theta + 1.$

4. $\cos 9\theta = 256\cos^9\theta - 576\cos^7\theta + 432\cos^5\theta - 120\cos^3\theta + 9\cos\theta.$

5. Obtain the series for $\cos n\theta$ in ascending powers of $\cos\theta$ directly from the equation

$$\frac{1-x^2}{1-2x\cos\theta+x^2} = 1 + 2x\cos\theta + 2x^2\cos 2\theta + \dots.$$

[Expand $(1-x^2)\{1+x(x-2\cos\theta)\}^{-1}$.]

6. Obtain the series for $\dfrac{\sin n\theta}{\sin \theta}$, in ascending powers of $\cos\theta$ directly from the equation

$$\frac{x\sin\theta}{1-2x\cos\theta+x^2}=\sin\theta+x\sin 2\theta+x^2\sin 3\theta+\dots.$$

7. Use the method of the note of § 7 to find the expansions for $\cosh nu$ and $\dfrac{\sinh nu}{\sinh u}$ corresponding to formulae (1) to (10).

8. Obtain the expansion for $\cos n\theta$ in descending powers of $\cos\theta$ from the equation

$$\frac{1-x\cos\theta}{1-2x\cos\theta+x^2}=1+x\cos\theta+x^2\cos 2\theta+\dots.$$

For examples on the applications of these expansions to roots of equations see Chapter XXIV, § 6.

§ 8. To express $\cos^n\theta$, $\sin^n\theta$ in a finite series of cosines or sines of multiples of θ (n being a positive integer).

Let
$$z=\cos\theta+i\sin\theta,$$
$$\therefore\ \frac{1}{z}=\cos\theta-i\sin\theta,$$

then $z^r+\dfrac{1}{z^r}=2\cos r\theta$, $z^r-\dfrac{1}{z^r}=2i\sin r\theta$, for all positive integral values of r.

$$\begin{aligned}(2\cos\theta)^n&=\left(z+\frac{1}{z}\right)^n\\ &=\left(z^n+\frac{1}{z^n}\right)+n\left(z^{n-2}+\frac{1}{z^{n-2}}\right)+\frac{n(n-1)}{\underline{|2}}\left(z^{n-4}+\frac{1}{z^{n-4}}\right)+\dots\\ &=2\cos n\theta+n\,2\cos(n-2)\theta+\frac{n(n-1)}{\underline{|2}}\,2\cos(n-4)\theta+\dots,\end{aligned}$$

therefore, when n is odd,

$$2^{n-1}\cos^n\theta=\cos n\theta+n\cos(n-2)\theta+\frac{n(n-1)}{\underline{|2}}\cos(n-4)\theta+\dots$$
$$+\frac{\underline{|n}}{\underline{|(n-1)/2}\ \underline{|(n+1)/2}}\cos\theta, \qquad (11)$$

and, when n is even,

$$2^{n-1}\cos^n\theta=\cos n\theta+n\cos(n-2)\theta+\frac{n(n-1)}{\underline{|2}}\cos(n-4)\theta+\dots$$
$$+\frac{1}{2}\frac{\underline{|n}}{\underline{|n/2}\ \underline{|n/2}}. \qquad (12)$$

In a similar way, from

$$(2i\sin\theta)^n = \left(z - \frac{1}{z}\right)^n,$$

we find that, when n is odd,

$$(-1)^{\frac{n-1}{2}} 2^{n-1} \sin^n\theta = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{\lfloor 2}\sin(n-4)\theta - \dots$$

$$+ (-1)^{\frac{n-1}{2}} \frac{\lfloor n}{\lfloor (n-1)/2 \; \lfloor (n+1)/2}\sin\theta, \qquad (13)$$

and, when n is even,

$$(-1)^{\frac{n}{2}} 2^{n-1} \sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{\lfloor 2}\cos(n-4)\theta - \dots$$

$$+ (-1)^{\frac{n}{2}} \frac{1}{2} \frac{\lfloor n}{\lfloor n/2 \; \lfloor n/2}\cos\theta. \qquad (14)$$

(13), (14) can also be obtained from (11), (12) by putting $\frac{\pi}{2} - \theta$ for θ.

The above method may also be used to express $\cos^p\theta \sin^q\theta$ in a series of multiples of sines and cosines of θ, where p and q are positive integers.

EXERCISE XX. b.

Prove, without quoting the results of § 8:

1. $2^3\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3.$
2. $2^4\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta.$
3. $2^5\sin^6\theta = -\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10.$
4. $2^6\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta.$
5. $2^6\sin^7\theta = -\sin 7\theta + 7\sin 5\theta - 21\sin 3\theta + 35\sin\theta.$
6. $2^7\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35.$
7. $2^7\sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35.$
8. $2^8\sin^9\theta = \sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta.$
9. $2^9\cos^{10}\theta = \cos 10\theta + 10\cos 8\theta + 45\cos 6\theta + 120\cos 4\theta + 210\cos 2\theta + 126.$
10. $2^6\cos^2\theta\sin^5\theta = \sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta.$
11. $2^6\cos^3\theta\sin^4\theta = \cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta.$
12. $2^9\cos^4\theta\sin^6\theta = -\cos 10\theta + 2\cos 8\theta + 3\cos 6\theta - 8\cos 4\theta - 2\cos 2\theta + 12.$

CHAPTER XXI

EXPANSIONS IN INFINITE SERIES

§ 1. Expansions in powers of the variable can often be obtained by Taylor's Theorem or Maclaurin's Theorem, but the differentiation may become heavy; other methods often give an expansion more easily, and expansions in series of sines or cosines of multiples of the variable must be obtained by other methods.

Various methods are given in Examples in this chapter. In Chapter XXIII it will be shown how many expansions can be obtained from factors by using logarithms.

§ 2. The following expansions were found, in Chap. XV, by a method which is practically the equivalent of using Maclaurin's Theorem

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \dots \textit{ ad infin.},$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \dots \textit{ ad infin.}$$

In Chap. XVIII, it was pointed out that both these series are convergent.

In § 3 we shall indicate how these expansions may be obtained without calculus.

The hyperbolic functions $\sinh x$ and $\cosh x$ are defined by the series (see Chap. XV, § 6, p. 254)

$$\sinh x = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \dots \textit{ ad infin.},$$

$$\cosh x = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \dots \textit{ ad infin.}$$

§ 3. The series for $\sin x$ and $\cos x$ can be found without the use of the calculus as follows.

In Chap. XIV, § 16, p. 244, it is proved† that, if n is a positive integer,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{\lfloor 2}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\cos^{n-4}\theta\sin^4\theta - \dots.$$

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† The proof given on p. 244 depends on de Moivre's Theorem; but it is pointed out on pp. 243, 244 that the expression quoted for $\cos n\theta$ can be obtained from the addition formulae, which can be obtained by mathematical induction, so that this expression for $\cos n\theta$ can be obtained without the use of complex numbers.

Put $n\theta = x$.

$$\therefore \cos x = \cos^n \frac{x}{n} - \frac{n(n-1)}{\lfloor 2} \cos^{n-2}\frac{x}{n}\sin^2\frac{x}{n} + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\cos^{n-4}\frac{x}{n}\sin^4\frac{x}{n} - \dots$$

$$= \cos^n\frac{x}{n} - \frac{x(x-\theta)}{\lfloor 2}\cos^{n-2}\frac{x}{n}\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2 + \frac{x(x-\theta)(x-2\theta)(x-3\theta)}{\lfloor 4}\cos^{n-4}\frac{x}{n}\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^4 - \dots.$$

Now let x remain finite, but let $n \to \infty$, so that $\theta \to 0$.

It is possible to deduce that

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \dots \textit{ ad infin.},$$

but to make the proof complete it is necessary to consider various limits (see § 4) and to discuss the question of convergence.

Similarly the equation

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{\lfloor 3}\cos^{n-3}\theta\sin^3\theta + \dots$$

leads to

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \dots \textit{ ad infin.}$$

§ 4. To find $\underset{n\to\infty}{\mathrm{Lt}} \cos\left(\frac{x}{n}\right)^n$.

Let $y = \left(\cos\frac{x}{n}\right)^n$.

Then $\log y = n\log\left(\cos\frac{x}{n}\right)$

$$= n\log\left(1 - 2\sin^2\frac{x}{2n}\right)$$

$$= n\left\{-2\sin^2\frac{x}{2n} - \frac{\left(2\sin^2\frac{x}{2n}\right)^2}{2} - \frac{\left(2\sin^2\frac{x}{2n}\right)^3}{3} - \dots\right\}$$

$$= -2n\sin\frac{x}{2n}\left\{\sin\frac{x}{2n}+\sin^3\frac{x}{2n}+\frac{2^2\sin^5\frac{x}{2n}}{3}+\dots\right\}$$

$$= -x\frac{\sin\frac{x}{2n}}{\frac{x}{2n}}\left\{\sin\frac{x}{2n}+\sin^3\frac{x}{2n}+\frac{2^2\sin^5\frac{x}{2n}}{3}+\dots\right\}.$$

$\therefore \underset{n\to\infty}{\mathrm{Lt}} \log y = -x\times 1\times$ (a factor which approaches zero)

$=0.$

$$\therefore \underset{n\to\infty}{\mathrm{Lt}}\left(\cos\frac{x}{n}\right)^n=1.$$

To find $\underset{n\to\infty}{\mathrm{Lt}}\left(\dfrac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^n$.

Since $\frac{x}{n}$ is acute, when n is large

$$\tan\frac{x}{n}>\frac{x}{n}>\sin\frac{x}{n}.$$

$$\therefore \frac{1}{\cos\frac{x}{n}}>\frac{\frac{x}{n}}{\sin\frac{x}{n}}>1,$$

$$\therefore \frac{1}{\left(\cos\frac{x}{n}\right)^n}>\left(\frac{\frac{x}{n}}{\sin\frac{x}{n}}\right)^n>1.$$

Now $\underset{n\to\infty}{\mathrm{Lt}}\left(\cos\frac{x}{n}\right)^n=1.$

$\therefore \left(\dfrac{\frac{x}{n}}{\sin\frac{x}{n}}\right)^n$ lies between 1 and a quantity which approaches 1 in the limit.

$$\therefore \underset{n\to\infty}{\mathrm{Lt}}\left(\frac{\frac{x}{n}}{\sin\frac{x}{n}}\right)^n=1, \qquad \therefore \underset{n\to\infty}{\mathrm{Lt}}\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^n=1.$$

§ 5. **Example i.** *Expand* $\tan x$ *in ascending powers of* x, *up to* x^5.

Since the series for $\sin x$ and $\cos x$ are both absolutely convergent for all values of x (see Chap. XVIII, § 10),

$$\begin{aligned}\tan x &= \frac{\sin x}{\cos x} = \sin x\,(\cos x)^{-1}\\ &= \left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \dots\right)\left\{1 - \left(\frac{x^2}{\lfloor 2} - \frac{x^4}{\lfloor 4} + \dots\right)\right\}^{-1}\\ &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)\left(1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + \dots\right)\\ &= x + \frac{x^3}{3} + \frac{2}{15}x^5, \text{ neglecting higher powers of } x.\end{aligned}$$

Example ii. *Find the expansion for* $\sin^{-1} x$ *in ascending powers of* x, *where* x *lies between* ± 1.

We know $\int_0^x \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$, where $\sin^{-1} x$ is the principal value.

By the binomial theorem, x being between ± 1,

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1\,.\,3}{2\,.\,4}x^4 + \dots\,.$$

Therefore, integrating each side,

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\frac{x^5}{5} + \dots \text{ (see Chap. XVIII, § 5)},$$

and this series is convergent, x being between ± 1.

EXERCISE XXI. a.

1. Use the series for $\sin x$ and $\cos x$ to find, correct to 4 decimal places, the sines and cosines of angles of (i) $\frac{1}{10}$ of a radian (ii) $\frac{1}{2}$ of a radian. Check by aid of four-figure tables.

2. If $\frac{\sin\theta}{\theta} = \frac{999}{1000}$, find θ to the nearest minute.

3. If $\cos\theta = 0{\cdot}9994$, find θ to the nearest minute.

4. If $\tan\theta = 0{\cdot}02$, find θ to the nearest minute.

5. If $\sin\theta = 0{\cdot}9998$, find θ to the nearest tenth of a minute.

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6. Find a series for $\cos^{-1} x$, by the method of § 5, Example ii.

7. Find the first three terms in the expansion of $\sinh^{-1} x$, when $x < 1$.

8. Prove that $\frac{1}{2}(\sinh\theta + \sin\theta) = \theta + \frac{\theta^5}{\lfloor 5} + \frac{\theta^9}{\lfloor 9} + \dots$

and $\frac{1}{2}(\cosh\theta + \cos\theta) = 1 + \frac{\theta^4}{\lfloor 4} + \frac{\theta^8}{\lfloor 8} + \dots\,.$

9. Prove that, when x is the circular measure of a small angle,

$$\cos^2 x = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \dots .$$

10. Prove that, when x is small,

$$\cos(a+x) = \cos a - x \sin a - \frac{x^2}{2}\cos a + \dots .$$

From the values of $\sin 60°$ and $\cos 60°$ deduce the value of $\cos 63°$ to 3 decimal places.

11. If $\cos^3\theta$ be expanded in a series of even powers of θ, prove that the coefficient of θ^{2n} in the expansion is $\dfrac{3(-1)^n}{4\,\underline{|2n}}(1+3^{2n-1})$.

12. Prove that $\cos x \cosh x = 1 - \dfrac{2^2 x^4}{\underline{|4}} + \dfrac{2^4 x^8}{\underline{|8}} - \dots .$

13. Find the first significant term in the expansion in ascending powers of θ of $2\theta - \dfrac{28\sin\theta + \sin 2\theta}{9 + 6\cos\theta}$.

14. An approximate value for the angle ϕ, measured in radians, is

$$\frac{3\sin\phi}{2+\cos\phi},$$

provided ϕ is less than $\dfrac{\pi}{2}$. Establish this result when ϕ is small, and show that the error is approximately $\dfrac{\phi^5}{180}$.

15. Find values of a and b which make $a\sin 2\theta - b\sin\theta$ approximately equal in value to θ, when θ is small.

16. Prove that $\sin\theta = \theta(\cos\theta)^{\frac{1}{3}}$ approximately. Deduce that, when θ is small,

$$\log\sin\theta = \log\theta + \tfrac{1}{3}\log\cos\theta.$$

17. Prove that, if powers of x above the fourth are neglected,

$$\log\frac{\sin x}{x} + \tfrac{1}{3}\log\sec x = \frac{x^4}{25} = \log\frac{\tan x}{x} - \tfrac{2}{3}\log\sec x.$$

18. Prove that the roots (other than zero) of the equation $\tan x = x$ are approximately $\pm\left(\dfrac{1}{a} - a - \dfrac{2a^3}{3}\right)$, where $\dfrac{1}{a} = (2n+1)\dfrac{\pi}{2}$ and n is any integer.

[From the graphs of $\tan x$ and x, it can be seen that the roots are near to $\pm\dfrac{1}{a}$.]

19. Evaluate: (i) $\underset{\theta\to 0}{\text{Lt}} \dfrac{2\sin\theta - \sin 2\theta}{\theta^3}$, (ii) $\underset{\theta\to 0}{\text{Lt}} \dfrac{\tan 2\theta - 2\sin\theta}{\theta^3}$,

(iii) $\underset{\theta\to 0}{\text{Lt}} \dfrac{\cos^2 a\theta - \cos^2 b\theta}{1-\cos c\theta}$, (iv) $\underset{\theta\to \frac{\pi}{2}}{\text{Lt}} (\sec\theta - \tan\theta)$,

(v) $\underset{n\to\infty}{\text{Lt}} \left(\cos\dfrac{a}{n}\right)^{n^2}$.

20. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{3\sinh x - 3\sin x - \tan^3 x}{x^5}$.

21. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{\tan(\sin x) - \sin(\tan x)}{\sin x - x\cos x - \frac{1}{3}\sin^3 x}$.

§ 6. The expansion of $\tan^{-1}\mathbf{x}$ ($\mathbf{x}$ between ± 1). Gregory's Series.

We know that $\displaystyle\int_0^x \frac{dx}{1+x^2} = \tan^{-1}x$, where $\tan^{-1}x$ is the principal value.

As $x^2 < 1$, $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots,$$
which is *absolutely* convergent for x between ± 1.

Therefore, integrating each side from 0 to x (see Chap. XVIII, § 5, p. 304),

$$\mathbf{\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots,}$$

when x lies between ± 1. The series is also convergent when $x = 1$, but not when $x = -1$.

It is interesting to see how the expansion may be obtained without calculus.

$$\log(1+ix) = \log\sqrt{1+x^2} + i\tan^{-1}x,$$

therefore, when x lies between ± 1,

$$\log\sqrt{1+x^2} + i\tan^{-1}x = ix - \frac{i^2x^2}{2} + \frac{i^3x^3}{3} - \ldots$$

and $$\log\sqrt{1+x^2} - i\tan^{-1}x = -ix - \frac{i^2x^2}{2} - \frac{i^3x^3}{3} - \dots .$$

$$\therefore\ 2i\tan^{-1}x = 2\left(ix + \frac{i^3x^3}{3} + \frac{i^5x^5}{5} + \dots\right),$$

$$\therefore\ \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots .$$

$\tan^{-1}x$ is the principal value and so lies between $\pm\frac{\pi}{4}$.

If θ is any one of the angles $\mathrm{Tan}^{-1}x$,

$$\theta - n\pi = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \dots,$$

where n is to be so chosen that $\theta - n\pi$ lies between $\pm\frac{\pi}{4}$.

Gregory's Series can be used to evaluate π.

When $x=1$, $\tan^{-1}x = \frac{\pi}{4}$,

$$\therefore\ \pi = 4\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \dots\right).$$

This series, however, converges very slowly. Various formulae have been devised to give series which converge more rapidly; the best known of these are given in Exercise XXI. b, No. 9.

EXERCISE XXI. b.

ON GREGORY'S SERIES.

1. Find the expansion for $\tanh^{-1}x$ in ascending powers of x. Are there any restrictions to the value of x?

2. Prove $\tan^{-1}x =$ Gregory's Series, by using the expansions for $\log(1+i\tan\theta)$ and for $\log(1-i\tan\theta)$.

3. Prove that, if θ lies between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$,

$$\theta = \frac{\pi}{2} - \cot\theta + \tfrac{1}{3}\cot^3\theta - \tfrac{1}{5}\cot^5\theta + \dots .$$

4. Prove that, when x lies between $\pm\frac{\pi}{4}$,

$$\tan x - \tfrac{1}{3}\tan^3x + \tfrac{1}{5}\tan^5x - \dots = \tanh x + \tfrac{1}{3}\tanh^3x - \tfrac{1}{5}\tanh^5x + \dots .$$

5. Sum the series $1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \dots$ *ad infin.*

6. Show that, if x lies between ± 1,

$$\log\left(\frac{\tan^{-1}x}{x}\right) = -\frac{x^2}{3} + \frac{13x^4}{90} - \frac{251x^6}{2835} + \dots .$$

7. Sum to infinity the series

$$x\cos\theta - \frac{x^3}{3}\cos 3\theta + \frac{x^5}{5}\cos 5\theta - \dots \quad (x<1).$$

8. Prove that, provided x is not an odd multiple of $\frac{\pi}{2}$,

$$\sin x - \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x - \dots = \tfrac{1}{2}(\sin x + \tfrac{1}{3}\sin^3 x + \tfrac{1}{5}\sin^5 x + \dots).$$

9. Prove the truth of the following identities which have been used for the evaluation of π:

(i) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$. (Euler.)

(ii) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$. (Dase.)

(iii) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$. (Machin.

(iv) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}$. (Rutherford.)

(v) $5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79} = \frac{\pi}{4}$. (Euler.)

§ 7. The two following examples, which are very alike, are worked out in full so that the reader may compare the two methods used.

Example iii. *Factorise $1 - 2x\cos\theta + x^2$, and hence obtain the expansion of $\log(1 - 2x\cos\theta + x^2)$ in a series of cosines of multiples of θ (x between ± 1).*

Deduce by differentiation that

$$x\sin\theta + x^2\sin 2\theta + x^3\sin 3\theta + \dots = \frac{x\sin\theta}{1 - 2x\cos\theta + x^2}.$$

$$1 - 2x\cos\theta + x^2 = \{1 - (x, \theta)\}\{1 - (x, -\theta)\}.$$

$$\therefore \log(1 - 2x\cos\theta + x^2) = \log\{1 - (x, \theta)\} + \log\{1 - (x, -\theta)\}$$

$$= -\left\{\frac{(x, \theta)}{1} + \frac{(x^2, 2\theta)}{2} + \frac{(x^3, 3\theta)}{3} + \dots\right.$$

$$\left. + \frac{(x, -\theta)}{1} + \frac{(x^2, -2\theta)}{2} + \frac{(x^3, -3\theta)}{3} + \dots\right\} \quad \text{(as } x \text{ lies between } \pm 1\text{)}.$$

$$\therefore \log(1 - 2x\cos\theta + x^2) = -2\left\{x\cos\theta + \frac{x^2}{2}\cos 2\theta + \frac{x^3}{3}\cos 3\theta + \dots\right\},$$

Differentiate each side of this equation with respect to θ (see Chap. XVIII, § 5),

$$\frac{2x\sin\theta}{1-2x\cos\theta+x^2}=2\{x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\ldots\},$$

which is convergent when x lies between ± 1,

$$\therefore\ x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\ldots=\frac{x\sin\theta}{1-2x\cos\theta+x^2}.$$

Example iv. *Prove that* (x *between* ± 1)

$$\tfrac{1}{2}\log(1+2x\cos\theta+x^2)=x\cos\theta-\tfrac{1}{2}x^2\cos 2\theta+\tfrac{1}{3}x^3\cos 3\theta-\ldots$$

and $$\tan^{-1}\frac{x\sin\theta}{1+2x\cos\theta+x^2}=x\sin\theta-\tfrac{1}{2}x^2\sin 2\theta+\tfrac{1}{3}x^3\sin 3\theta-\ldots.$$

Since x is numerically <1,

$$\log\{1+(x,\theta)\}=(x,\theta)-\tfrac{1}{2}(x^2,2\theta)+\tfrac{1}{3}(x^3,3\theta)-\ldots. \qquad (1)$$

But $1+(x,\theta)=1+x\cos\theta+ix\sin\theta$

$$=\sqrt{1+2x\cos\theta+x^2}\,(\cos\phi+i\sin\phi),\text{ where }\phi=\tan^{-1}\frac{x\sin\theta}{1+x\cos\theta}.$$

$\therefore\ \log\{1+(x,\theta)\}=\tfrac{1}{2}\log(1+2x\cos\theta+x^2)+i\tan^{-1}\dfrac{x\sin\theta}{1+x\cos\theta}$, as $1+x\cos\theta$ is positive (see Chap. XVII, § 10).

Therefore, equating the real parts of (1),

$$\tfrac{1}{2}\log(1+2x\cos\theta+x^2)=x\cos\theta-\tfrac{1}{2}x^2\cos 2\theta+\tfrac{1}{3}x^3\cos 3\theta-\ldots$$

(Note that the result of Example iii follows from this by changing x to $-x$) and, equating the imaginary parts of (1),

$$\tan^{-1}\frac{x\sin\theta}{1+x\cos\theta}=x\sin\theta-\frac{x^2}{2}\sin 2\theta+\frac{x^3}{3}\sin 3\theta-\ldots.$$

NOTE. Both series can be proved to be convergent when $x=1$, provided θ is not an odd multiple of π.

$$\therefore\ \tfrac{1}{2}\log 2(1+\cos\theta)=\cos\theta-\tfrac{1}{2}\cos 2\theta+\tfrac{1}{3}\cos 3\theta-\ldots,$$

$$\text{i.e.}\quad \log\left(2\cos\frac{\theta}{2}\right)=\cos\theta-\tfrac{1}{2}\cos 2\theta+\tfrac{1}{3}\cos 3\theta-\ldots$$

and $$\tan^{-1}\frac{\sin\theta}{1+\cos\theta}=\sin\theta-\tfrac{1}{2}\sin 2\theta+\tfrac{1}{3}\sin 3\theta-\ldots.$$

Now $$\frac{\sin\theta}{1+\cos\theta}=\tan\frac{\theta}{2},$$

$$\therefore\ \frac{\theta}{2}=\sin\theta-\tfrac{1}{2}\sin 2\theta+\tfrac{1}{3}\sin 3\theta-\ldots,$$

where θ lies between $\pm\pi$.

§8. **Example v.** *Expand $e^{ax}\sin bx$ in a series of ascending powers of x.*

$$e^{ax}\sin bx=\frac{1}{2i}\exp(ax)\{\exp(ibx)-\exp(-ibx)\}$$

$$=\frac{1}{2i}\{\exp(a+ib)x-\exp(a-ib)x\}.$$

Let $a+ib=(r,\,\theta)$.

Then $e^{ax}\sin bx=\frac{1}{2i}\{\exp(xr,\,\theta)-\exp(xr,\,-\theta)\}$

$$=\frac{1}{2i}\left\{1+\frac{(xr,\,\theta)}{\lfloor 1}+\frac{(x^2r^2,\,2\theta)}{\lfloor 2}+\ldots\right.$$

$$\left.-1-\frac{(xr,\,-\theta)}{\lfloor 1}-\frac{(x^2r^2,\,-2\theta)}{\lfloor 2}-\ldots\right\}$$

$$=\frac{1}{2i}\left\{\frac{2xr\,i\sin\theta}{\lfloor 1}+\frac{2x^2r^2\,i\sin 2\theta}{\lfloor 2}+\ldots\right\}.$$

$$\therefore\ e^{ax}\sin bx=\frac{xr\sin\theta}{\lfloor 1}+\frac{x^2r^2\sin 2\theta}{\lfloor 2}+\frac{x^3r^3\sin 3\theta}{\lfloor 3}+\ldots,$$

where $r=\sqrt{a^2+b^2}$ and θ is such that $\sin\theta=\frac{b}{\sqrt{a^2+b^2}}$, $\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$.

Example vi. *If $\sin x=h\sin(x+a)$, where $-1<h<1$, show that*

$$x=n\pi+h\sin a+\frac{h^2}{2}\sin 2a+\frac{h^3}{3}\sin 3a+\ldots,$$

where n is any integer.

$$\sin x=h\sin(x+a)=h(\sin x\cos a+\cos x\sin a).$$

$$\therefore\ \tan x=\frac{h\sin a}{1-h\cos a}.$$

$$\therefore\ \frac{\exp(ix)-\exp(-ix)}{\exp(ix)+\exp(-ix)}=\frac{ih\sin a}{1-h\cos a},$$

$$\therefore\ \frac{\exp(ix)}{\exp(-ix)}=\frac{1-h\cos a+ih\sin a}{1-h\cos a-ih\sin a}.$$

$$\therefore\ \exp(2ix)=\frac{1-(h,\,-a)}{1-(h,\,a)}.$$

$$\therefore\ 2n\pi i+2ix=\log\{1-(h,\,-a)\}-\log\{1-(h,\,a)\}$$

$$=-(h,\,-a)-\tfrac{1}{2}(h^2,\,-2a)-\tfrac{1}{3}(h^3,\,-3a)-\ldots$$

$$+(h,\,a)+\tfrac{1}{2}(h^2,\,2a)+\tfrac{1}{3}(h^3,\,3a)+\ldots\ (\text{since } -1<h<1)$$

$$=h2i\sin a+\frac{h^2}{2}2i\sin 2a+\frac{h^3}{3}2i\sin 3a+\ldots.$$

$$\therefore\ x=n\pi+h\sin a+\frac{h^2}{2}\sin 2a+\frac{h^3}{3}\sin 3a+\ldots,\text{ where } n \text{ is any integer.}$$

EXERCISE XXI. c.

1. Expand $\log(1-2x\cos\theta+x^2)$ in ascending powers of x, when $x<1$; and deduce that

$$\frac{\cos\theta-x}{1-2x\cos\theta+x^2}=\cos\theta+x\cos 2\theta+x^2\cos 3\theta+\dots.$$

2. Expand in ascending powers of x $(x<1)$:

(i) $\dfrac{1+x\cos\theta}{1+2x\cos\theta+x^2}$, (iii) $\dfrac{1-x^2}{1-2x\cos\theta+x^2}$,

(ii) $\dfrac{2x\cos\theta}{1-2x\sin\theta+x^2}$, (iv) $\dfrac{1}{1-2x\cos\theta+x^2}$.

3. Prove that the coefficient of x^n in the expansion in ascending powers of x of $\dfrac{1-x}{1-2x\cos\theta+x^2}$ is $\cos\left(n+\frac{1}{2}\right)\theta\sec\frac{1}{2}\theta$.

4. Expand in ascending powers of x:

(i) $e^{x\cos\theta}\cos(x\sin\theta)$, (ii) $e^{ax}\cos bx$.

5. Expand $\log(\sec\theta+\tan\theta)$ in ascending powers of $\sin\theta$.

6. Expand $\log\cos\theta$ in ascending powers of $\tan\dfrac{\theta}{2}$, where θ is an acute angle.

7. Expand $\log\cos\theta$ in ascending powers of $\tan\theta$, where θ lies between $\pm\dfrac{\pi}{4}$.

8. Prove that, for an acute angle θ,

$$\log 2\cos\theta=\cos 2\theta-\tfrac{1}{2}\cos 4\theta+\tfrac{1}{3}\cos 6\theta-\dots.$$

9. Prove that, for an acute angle θ,

$$\log 2\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=-\sin\theta+\tfrac{1}{2}\cos 2\theta+\tfrac{1}{3}\sin 3\theta-\tfrac{1}{4}\cos 4\theta-\tfrac{1}{5}\sin 5\theta+\dots.$$

10. Prove that, for an acute angle θ,

$$\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=2\left\{\sin\theta-\tfrac{1}{3}\sin 3\theta+\tfrac{1}{5}\sin 5\theta-\dots\right\}.$$

11. Expand $\tan^{-1}\dfrac{r\sin\theta}{1-r\cos\theta}$ in powers of r, where $r<1$.

12. If $\sin x=y\cos(x+a)$, expand x in ascending powers of y. $(-1<y<1.)$

13. Prove that, if $\tan x=3\tan(x-y)$,

$$y=\frac{1}{1.2}\sin 2x-\frac{1}{2.2^2}\sin 4x+\frac{1}{3.2^3}\sin 6x-\dots.$$

14. If $\tan\alpha=\cos 2\omega\tan\lambda$, prove that

$$\lambda-\alpha=\tan^2\omega\sin 2\alpha+\tfrac{1}{2}\tan^4\omega\sin 4\alpha+\tfrac{1}{3}\tan^6\omega\sin 6\alpha+\dots.$$

15. If $\tan\frac{\theta}{2}=2\tan\frac{a}{2}$, prove that

$$\tfrac{1}{2}(\theta-a)=\tan^{-1}\frac{\sin a}{3-\cos a}=\frac{1}{3}\sin a+\frac{1}{2.3^2}\sin 2a+\frac{1}{3.3^3}\sin 3a+\dots.$$

16. Show that, when ϕ is positive,

$$\frac{\sin\theta}{\cosh\phi-\cos\theta}=2\sum_{n=1}^{n=\infty}e^{-n\phi}\sin n\theta.$$

17. If B is an angle of a triangle ABC and less than A, prove that

$$B=\frac{b}{a}\sin C+\frac{1}{2}\frac{b^2}{a^2}\sin 2C+\frac{1}{3}\frac{b^3}{a^3}\sin 3C+\dots.$$

18. Prove that the coefficient of x^n, in the expansion in ascending powers of x, of $e^{x\cos a}\sin(a+x\sin\beta)$ is $\frac{r^n}{\underline{|n}}\sin(n\theta+a)$, where $\cos a+i\sin\beta=(r,\theta)$.

19. Prove that the coefficient of x^n in the expansion of $\dfrac{a}{ax^2-2bx+c}$ in ascending powers of x, where $ac>b^2$, is $\left(\dfrac{a}{c}\right)^{\frac{n+2}{2}}\dfrac{\sin(n+1)\theta}{\sin\theta}$, where $\sqrt{ac}\cos\theta=b$.

20. Expand $e^x\cos x$ in ascending powers of x; and show, by equating the coefficients of x^n in the expansions of $e^x\sin x\times e^x\cos x$ and of $\frac{1}{2}e^{2x}\sin 2x$, that

$$n\sin\frac{(n-1)\pi}{4}\cos\frac{\pi}{4}+\frac{n(n-1)}{1.2}\sin\frac{(n-2)\pi}{4}\cos\frac{2\pi}{4}+\dots+n\sin\frac{\pi}{4}\cos\frac{(n-1)\pi}{4}$$

$$=(2^{n-1}-1)\sin\frac{n\pi}{4}.$$

CHAPTER XXII

FACTORISATION. FINITE PRODUCTS

§ 1. There are various methods of factorisation, but the most useful method applicable to trigonometrical functions is to deduce factors from consideration of the roots of equations.

This method is applied to various functions in the first part of the chapter, after that many forms which can be deduced from these fundamental factors are considered.

Most of the factor forms can be derived from the general factor theorem of § 9.

§ 2. Factorisation of $\dfrac{\sin n\theta}{\sin\theta}$, where n is an integer.

In Chap. XIV, § 16, p. 245, it was shown that $\dfrac{\sin n\theta}{\sin\theta}$ can be expressed in a series of powers of $\cos\theta$, the term of highest degree being $2^{n-1}\cos^{n-1}\theta$.

$\therefore$ $\dfrac{\sin n\theta}{\sin\theta}$ can be expressed as the product of $\overline{n-1}$ factors thus:

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1}(\cos\theta - \quad)(\cos\theta - \quad)(\cos\theta - \quad)\ldots.$$

Now $\sin n\theta = 0 = \sin p\pi$, where p is zero or any integer, when

$$\theta = 0,\ \frac{\pi}{n},\ \frac{2\pi}{n},\ \frac{3\pi}{n},\ \ldots,\ \frac{\overline{n-1}\,\pi}{n},$$

$$\therefore\ \sin n\theta = 0 \text{ when } \cos\theta = \cos\frac{\pi}{n},\ \cos\frac{2\pi}{n},\ \cos\frac{3\pi}{n},\ \ldots,\ \cos\frac{\overline{n-1}\,\pi}{n}.$$

The case of $p=0$ has been left out because that corresponds to $\sin\theta$ being a factor of $\sin n\theta$.

Now these values for $\cos\theta$ are all different, and no new values could be obtained by giving further integral values to p.

$$\therefore \frac{\sin n\theta}{\sin\theta} = 2^{n-1}\left(\cos\theta - \cos\frac{\pi}{n}\right)\left(\cos\theta - \cos\frac{2\pi}{n}\right)\dots\left(\cos\theta - \cos\frac{\overline{n-1}\,\pi}{n}\right).$$

This we shall abbreviate thus:

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1}\prod_{p=1}^{p=n-1}\left(\cos\theta - \cos\frac{p\pi}{n}\right). \qquad (1)$$

(*range of angles* π)

§ 3. By a similar method it can be shown that

$$\mathbf{cos}\, n\theta = 2^{n-1}\left(\cos\theta - \cos\frac{\pi}{2n}\right)\left(\cos\theta - \cos\frac{3\pi}{2n}\right)\dots\left(\cos\theta - \cos\frac{\overline{2n-1}\,\pi}{2n}\right)$$

$$= 2^{n-1}\prod_{p=1}^{p=n}\left(\cos\theta - \cos\frac{\overline{2p-1}\,\pi}{2n}\right) \qquad (2)$$

(*range* π)

§ 4. Factorisation of $\mathbf{cos}\, n\theta - \mathbf{cos}\, n\alpha$, where n is an integer.

In Chap. XIV, § 16, p. 244, it was shown that $\cos n\theta$ can be expressed in a series of powers of $\cos\theta$, the term of highest degree being $2^{n-1}\cos^n\theta$.

$\therefore$ $\cos n\theta - \cos n\alpha$ can be expressed as the product of n factors thus:

$$\cos n\theta - \cos n\alpha = 2^{n-1}(\cos\theta - \quad)(\cos\theta - \quad)(\cos\theta - \quad)\dots.$$

Now $\cos n\theta - \cos n\alpha = 0$, *i.e.* $\cos n\theta = \cos(n\alpha + 2p\pi)$, where p is zero or any integer, when

$$\theta = \alpha,\ \alpha + \frac{2\pi}{n},\quad \alpha + \frac{4\pi}{n},\quad \alpha + \frac{6\pi}{n},\quad \dots,\quad \alpha + \frac{2(n-1)\pi}{n}.$$

$\therefore$ $\cos n\theta - \cos n\alpha = 0$ when

$$\cos\theta = \cos\alpha,\ \cos\left(\alpha + \frac{2\pi}{n}\right),\ \cos\left(\alpha + \frac{4\pi}{n}\right),\ \dots,\ \cos\left(\alpha + \frac{\overline{2n-1}\,\pi}{n}\right)$$

Now these values for $\cos\theta$ are all different,† and no new values could be obtained by giving further integral values to p.

$$\therefore\ \cos n\theta - \cos n\alpha = 2^{n-1}[\cos\theta - \cos\alpha]\left[\cos\theta - \cos\left(\alpha + \frac{2\pi}{n}\right)\right]$$

$$\times\left[\cos\theta - \cos\left(\alpha + \frac{4\pi}{n}\right)\right]\dots\left[\cos\theta - \cos\left(\alpha + \frac{\overline{2n-1}\,\pi}{n}\right)\right]$$

$$= 2^{n-1}\prod_{p=0}^{p=n-1}\left[\cos\theta - \cos\left(\alpha + \frac{2p\pi}{n}\right)\right]. \qquad (3)$$

(range 2π)

In particular, put $n\alpha = \frac{\pi}{2}$, then

$$\cos n\theta = 2^{n-1}\prod_{p=0}^{p=n-1}\left[\cos\theta - \cos\left(\frac{\pi}{2n} + \frac{2p\pi}{n}\right)\right],$$

(range 2π)

which can be thrown into the same form as formula (2).

§ 5. Factorisation of $x^n - 1$, where n is an integer.

$x^n - 1 = 0$ when

$x^n = \cos 2p\pi + i\sin 2p\pi = (1,\ 2p\pi)$, where p is zero or any integer.

$\therefore\ x^n - 1 = 0$ when $x = \left(1,\ \frac{2p\pi}{n}\right)$, where $p = 0, 1, 2, \dots, \overline{n-1}$.

These n values of x are all different and no additional values for x could be obtained by giving further integral values to p.

$$\therefore\ x^n - 1 = \prod_{p=0}^{p=n-1}\left[x - \left(1,\ \frac{2p\pi}{n}\right)\right] \text{ or } \prod_{p=0}^{p=n-1}\left(x - \cos\frac{2p\pi}{n} - i\sin\frac{2p\pi}{n}\right).$$

This form has the disadvantage that most of the factors contain imaginary numbers, but we can group the factors in such a way that we get rid of the imaginary numbers.

---

† Except when $n\alpha$ is an odd multiple of π.

Leave the factor for which $p=0$ by itself, but group together the factors for which $p=1$ and $\overline{n-1}$, $p=2$ and $\overline{n-2}$, ...

$$\therefore\ x^n-1=(x-1)\left[x-\left(1,\ \frac{2\pi}{n}\right)\right]\left[x-1,\ \frac{\overline{2n-1}\,\pi}{n}\right]\left[x-\left(1,\ \frac{2\,.\,2\pi}{n}\right)\right]$$

$$\times\left[x-\left(1,\ \frac{\overline{2n-2}\,\pi}{n}\right)\right]\cdots$$

$$=(x-1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{2\,.\,2\pi}{n}+1\right)\cdots.$$

If n is odd, the last factor will be $\left(x^2-2x\cos\frac{n-1}{2}\frac{2\pi}{n}+1\right)$ and

$$x^n-1=(x-1)\prod_{p=1}^{p=\frac{n-1}{2}}\left(x^2-2x\cos\frac{p\,.\,2\pi}{n}+1\right).\qquad(4a)$$

(range π)

If n is even, a factor $\left[x-\left(1,\ \frac{n}{2}\frac{2\pi}{n}\right)\right]$ will remain over, and this s equal to $x+1$.

$$\therefore\ x^n-1=(x^2-1)\prod_{p=1}^{p=\frac{n}{2}-1}\left[x^2-2x\cos\frac{p\,.\,2\pi}{n}+1\right].\qquad(4b)$$

(range π)

§6. Factorisation of $\mathbf{x^n+1}$, where n is an integer.

By a method similar to that of § 5 we can factorise x^n+1.

$$x^n+1=0,\ \text{when } x^n=-1=(1,\ \pi+2p\pi),$$

$$\textit{i.e.}\text{ when } x=\left(1,\ \frac{\overline{2p+1}\,\pi}{n}\right),\ \text{where } p=0,1,2,\ldots,\overline{n-1}.$$

These n values are all different and no additional values for x could be obtained by giving further integral values to p.

$$\therefore\ x^n+1=\prod_{p=0}^{p=n-1}\left[x-\left(1,\ \frac{\overline{2p+1}\,\pi}{n}\right)\right].$$

By grouping the factors in pairs as in § 5, we find:
when n is odd

$$x^n + 1 = (x+1) \prod_{p=0}^{p=\frac{n-3}{2}} \left(x^2 - 2x\cos\frac{\overline{2p+1}\,\pi}{n} + 1\right), \qquad (5a)$$

(*range* π)

when n is even

$$x^n + 1 = \prod_{p=0}^{p=\frac{n-2}{2}} \left(x^2 - 2x\cos\frac{\overline{2p+1}\,\pi}{n} + 1\right). \qquad (5b)$$

(*range* π)

§ 7. Factorisation of

$$\mathbf{x^n + \frac{1}{x^n} - 2\cos n\alpha} \quad \text{or} \quad \mathbf{x^{2n} - 2x^n\cos n\alpha + 1},$$

where n is an integer.

Put $x^{2n} - 2x^n\cos n\alpha + 1 = 0$.

$$\therefore\ (x^n - \overline{\cos n\alpha + i\sin n\alpha})(x^n - \overline{\cos n\alpha - i\sin n\alpha}) = 0,$$

$$\therefore\ x^n = \cos n\alpha \pm i\sin n\alpha = \cos(n\alpha + 2p\pi) \pm i\sin(n\alpha + 2p\pi),$$

where p is zero or any integer,

$$\therefore\ x = \cos\left(\alpha + \frac{2p\pi}{n}\right) \pm i\sin\left(\alpha + \frac{2p\pi}{n}\right).$$

Now if $p = 0, 1, 2, \ldots, \overline{n-1}$, we get $2n$ different† values for x and we get no additional values by giving further values to p,

$$\therefore\ x^{2n} - 2x^n\cos n\alpha + 1 = \prod_{p=0}^{p=n-1} \left[x - \cos\left(\alpha + \frac{2p\pi}{n}\right) - i\sin\left(\alpha + \frac{2p\pi}{n}\right)\right]$$

$$\times \left[x - \cos\left(\alpha + \frac{2p\pi}{n}\right) + i\sin\left(\alpha + \frac{2p\pi}{n}\right)\right]$$

$$= \prod_{p=0}^{p=n-1} \left[x^2 - 2x\cos\left(\alpha + \frac{2p\pi}{n}\right) + 1\right]. \qquad (6a)$$

(*range* 2π)

† Except when $n\alpha$ is an odd multiple of π.

And $$x^n + \frac{1}{x^n} - 2\cos na = \prod_{p=0}^{p=n-1}\left[x + \frac{1}{x} - 2\cos\left(a + \frac{2p\pi}{n}\right)\right]. \qquad (6b)$$

Note. The factors of $\cos n\theta - \cos na$ (see § 4) may be deduced from the above thus:

Let $x = (1, \theta)$, then

$$\frac{1}{x} = (1, -\theta), \quad x^n = (1, n\theta), \quad \frac{1}{x^n} = (1, -n\theta).$$

$$\therefore\ x + \frac{1}{x} = 2\cos\theta, \quad x^n + \frac{1}{x^n} = 2\cos n\theta.$$

On substituting these values in (6*b*), we get (3) of § 4.

§ 8. Factorisation of **cosh nu − cos na**, where n is an integer.

From § 7

$$x^n + \frac{1}{x^n} - 2\cos na = \prod_{p=0}^{p=n-1}\left[x + \frac{1}{x} - 2\cos\left(a + \frac{2p\pi}{n}\right)\right].$$

Let $x = e^u$, then $\frac{1}{x} = e^{-u}$, $x^n = e^{nu}$, $\frac{1}{x^n} = e^{-nu}$.

$$\therefore\ x + \frac{1}{x} = 2\cosh u, \quad x^n + \frac{1}{x^n} = 2\cosh nu.$$

$$\therefore\ \cosh nu - \cos na = 2^{n-1}\prod_{p=0}^{p=n-1}\left\{\cosh u - \cos\left(a + \frac{2p\pi}{n}\right)\right\}. \qquad (7)$$

(*range* 2π)

§ 9. The results of §§ 4, 7, 8 may be stated thus:

$$2\cos n\theta - 2\cos na = \prod_{p=0}^{p=n-1}\left[2\cos\theta - 2\cos\left(a + \frac{2p\pi}{n}\right)\right],$$

$$x^n + \frac{1}{x^n} - 2\cos na = \prod_{p=0}^{p=n-1}\left[x + \frac{1}{x} - 2\cos\left(a + \frac{2p\pi}{n}\right)\right],$$

$$2\cosh nu - 2\cos na = \prod_{p=0}^{p=n-1}\left[2\cosh u - 2\cos\left(a + \frac{2p\pi}{n}\right)\right].$$

Thus we see that if v_n denote any of the functions

$$2\cos n\theta, \quad 2\cosh nu, \quad x^n + \frac{1}{x^n},$$

$$v_n - 2\cos na = \prod_{p=0}^{p=n-1}\left[v_1 - 2\cos\left(a + \frac{2p\pi}{n}\right)\right]. \qquad (8)$$

(*range* 2π)

This result, which we shall call the **General Factor Theorem**, is a convenient summary of the results proved in §§ 4, 7, 8.

§ 10. Various factor forms for $\dfrac{\sin n\theta}{\sin\theta}$.

From § 2

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\left(\cos\theta-\cos\frac{\pi}{n}\right)\left(\cos\theta-\cos\frac{2\pi}{n}\right)\ldots\left(\cos\theta-\cos\frac{\overline{n-1}\,\pi}{n}\right).$$

Now $$\cos\frac{\overline{n-p}\,\pi}{n}=-\cos\frac{p\pi}{n}.$$

Hence, taking together factors equidistant from the ends, we get: *when n is odd*

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\left(\cos^2\theta-\cos^2\frac{\pi}{n}\right)\left(\cos^2\theta-\cos^2\frac{2\pi}{n}\right)\ldots$$
$$\ldots\left(\cos^2\theta-\cos^2\frac{n-1}{2}\frac{\pi}{n}\right)\qquad(9a)$$
$$=2^{n-1}\left(\sin^2\frac{\pi}{n}-\sin^2\theta\right)\left(\sin^2\frac{2\pi}{n}-\sin^2\theta\right)\ldots$$
$$\ldots\left(\sin^2\frac{n-1}{2}\frac{\pi}{n}-\sin^2\theta\right);\qquad(10a)$$

when n is even, there is one middle factor $\left(\cos\theta-\cos\dfrac{n}{2}\dfrac{\pi}{n}\right)=\cos\theta$,

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\cos\theta\left(\cos^2\theta-\cos^2\frac{\pi}{n}\right)\left(\cos^2\theta-\cos^2\frac{2\pi}{n}\right)\ldots$$
$$\ldots\left(\cos^2\theta-\cos^2\frac{n-2}{2}\frac{\pi}{n}\right)\qquad(9b)$$
$$=2^{n-1}\cos\theta\left(\sin^2\frac{\pi}{n}-\sin^2\theta\right)\left(\sin^2\frac{2\pi}{n}-\sin^2\theta\right)\ldots$$
$$\ldots\left(\sin^2\frac{n-2}{2}\frac{\pi}{n}-\sin^2\theta\right).\qquad(10b)$$

If in (10a) and (10b) we make $\theta=0$, since $\underset{\theta\to0}{\mathrm{Lt}}\dfrac{\sin n\theta}{\sin\theta}=n$ (see Ex. IX. c, No. 4),

$$n=2^{n-1}\sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{n}\ldots,\qquad(11)$$

the last factor being $\sin^2\dfrac{n-1}{2}\dfrac{\pi}{n}$ or $\sin^2\dfrac{n-2}{2}\dfrac{\pi}{n}$ according as n is odd or even.

Hence, on dividing (10*a*) and (10*b*) by (11), we get;
when n is odd

$$\frac{\sin n\theta}{n \sin \theta} = \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi}{n}}\right)\left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{2\pi}{n}}\right) \cdots \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{n-1}{2}\frac{\pi}{n}}\right), \quad (12a)$$

when n is even

$$\frac{\sin n\theta}{n \sin \theta \cos \theta} = \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi}{n}}\right)\left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{2\pi}{n}}\right) \cdots \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{n-2}{2}\frac{\pi}{n}}\right). \quad (12b)$$

Note that the range of angles in formulae (9)–(12) is $\frac{\pi}{2}$ in each case.

For other forms see Exercise XXII. a, No. 6.

§ 11. Various forms for cos $n\theta$.

Starting from (2) in § 3 and proceeding as in § 10, we get:

when n is odd

$$\cos n\theta = 2^{n-1} \cos \theta \left(\cos^2 \theta - \cos^2 \frac{\pi}{2n}\right)\left(\cos^2 \theta - \cos^2 \frac{3\pi}{2n}\right) \cdots$$

$$\cdots \left(\cos^2 \theta - \cos^2 \frac{\overline{n-2}\,\pi}{2n}\right) \quad (13a)$$

$$= 2^{n-1} \cos \theta \left(\sin^2 \frac{\pi}{2n} - \sin^2 \theta\right)\left(\sin^2 \frac{3\pi}{2n} - \sin^2 \theta\right) \cdots$$

$$\cdots \left(\sin^2 \frac{\overline{n-2}\,\pi}{2n} - \sin^2 \theta\right), \quad (14a)$$

when n is even

$$\cos n\theta = 2^{n-1} \left(\cos^2 \theta - \cos^2 \frac{\pi}{2n}\right)\left(\cos^2 \theta - \cos^2 \frac{3\pi}{2n}\right) \cdots$$

$$\cdots \left(\cos^2 \theta - \cos^2 \frac{\overline{n-1}\,\pi}{2n}\right) \quad (13b)$$

$$= 2^{n-1} \left(\sin^2 \frac{\pi}{2n} - \sin^2 \theta\right)\left(\sin^2 \frac{3\pi}{2n} - \sin^2 \theta\right) \cdots$$

$$\cdots \left(\sin^2 \frac{\overline{n-1}\,\pi}{2n} - \sin^2 \theta\right). \quad (14b)$$

On putting $\theta = 0$,

$$1 = 2^{n-1} \sin^2 \frac{\pi}{n} \sin^2 \frac{3\pi}{n} \sin^2 \frac{5\pi}{n} \ldots, \tag{15}$$

the last factor being $\sin^2 \frac{\overline{n-2}\,\pi}{2n}$ or $\sin^2 \frac{\overline{n-1}\,\pi}{2n}$, according as n is odd or even.

Lastly, on dividing (14a) and (14b) by (15), we get:

when n is odd

$$\cos n\theta = \cos\theta \left(1 - \frac{\sin^2\theta}{\sin^2 \frac{\pi}{2n}}\right)\left(1 - \frac{\sin^2\theta}{\sin^2 \frac{3\pi}{2n}}\right) \cdots \left(1 - \frac{\sin^2\theta}{\sin^2 \frac{\overline{n-2}\,\pi}{2n}}\right), \tag{16a}$$

when n is even

$$\cos n\theta = \left(1 - \frac{\sin^2\theta}{\sin^2 \frac{\pi}{2n}}\right)\left(1 - \frac{\sin^2\theta}{\sin^2 \frac{3\pi}{2n}}\right) \cdots \left(1 - \frac{\sin^2\theta}{\sin^2 \frac{\overline{n-1}\,\pi}{2n}}\right). \tag{16b}$$

Note that the range of angles in formulae (13)–(16) is $\frac{\pi}{2}$ in each case.

For other forms see Exercise XXII. a, No. 7.

§ 12. The following method of finding some of the formulae of §§ 10, 11, taken substantially from Serret's *Trigonometry*, is of interest.

From Chap. XIV, § 16, p. 244,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{\lfloor 2} \cos^{n-2}\theta \sin^2\theta + \ldots,$$

$$\sin n\theta = \frac{n}{\lfloor 1} \cos^{n-1}\theta \sin\theta - \frac{n(n-1)(n-2)}{\lfloor 3} \cos^{n-3}\theta \sin^3\theta + \ldots.$$

If n is even

$$\cos n\theta = (1 - \sin^2\theta)^{\frac{n}{2}} - \frac{n(n-1)}{\lfloor 2} (1 - \sin^2\theta)^{\frac{n-2}{2}} \sin^2\theta + \ldots,$$

$$\sin n\theta = \cos\theta \left\{ \frac{n}{\lfloor 1} (1 - \sin^2\theta)^{\frac{n-2}{2}} \sin\theta \right.$$

$$\left. - \frac{n(n-1)(n-2)}{\lfloor 3} (1 - \sin^2\theta)^{\frac{n-4}{2}} \sin^3\theta + \ldots \right\}.$$

$\therefore\ \cos n\theta$ and $\dfrac{\sin n\theta}{n\sin\theta\cos\theta}$ are functions entirely of $\sin\theta$, which each reduce to 1 if $\sin\theta=0$.

Now $\cos n\theta$ is of degree n and vanishes if

$$\theta=-\frac{(n-1)\pi}{2n},-\frac{(n-3)\pi}{2n},\ \ldots,-\frac{3\pi}{2n},-\frac{\pi}{2n},\frac{\pi}{2n},\frac{3\pi}{2n},\ \ldots,\ \frac{(n-1)\pi}{2n},$$

$\therefore\ \cos n\theta$ is divisible by

$$1-\frac{\sin^2\theta}{\sin^2\frac{\pi}{2n}},\quad 1-\frac{\sin^2\theta}{\sin^2\frac{3\pi}{2n}},\quad \ldots,\quad 1-\frac{\sin^2\theta}{\sin^2\frac{(n-1)\pi}{2n}},$$

$\therefore$ as $\cos n\theta=1$ when $\theta=0$

$$\cos n\theta=\prod_{p=1}^{p=\frac{n}{2}}\left(1-\frac{\sin^2\theta}{\sin^2\frac{(2p-1)\pi}{2n}}\right). \qquad (16b)$$

Also $\dfrac{\sin n\theta}{n\sin\theta\cos\theta}$ is of degree $n-2$ and vanishes if

$$\theta=-\frac{(n-2)\pi}{2n},-\frac{(n-4)\pi}{2n},\ \ldots,-\frac{4\pi}{2n},-\frac{2\pi}{2n},\frac{2\pi}{2n},\frac{4\pi}{2n},\ \ldots,\frac{(n-2)\pi}{2n},$$

$\therefore\ \dfrac{\sin n\theta}{n\sin\theta\cos\theta}$ is divisible by

$$1-\frac{\sin^2\theta}{\sin^2\frac{2\pi}{2n}},\quad 1-\frac{\sin^2\theta}{\sin^2\frac{4\pi}{2n}},\quad \ldots,\quad 1-\frac{\sin^2\theta}{\sin^2\frac{(n-2)\pi}{2n}},$$

$\therefore$ as $\dfrac{\sin n\theta}{n\sin\theta}=1$ when $\theta=0$

$$\frac{\sin n\theta}{\sin\theta}=n\cos\theta\prod_{p=1}^{p=\frac{n-2}{2}}\left(1-\frac{\sin^2\theta}{\sin^2\frac{p\pi}{n}}\right). \qquad (12b)$$

Using the transformation $1-\dfrac{\sin^2 u}{\sin^2 v}=\cos^2 u\left(1-\dfrac{\tan^2 u}{\tan^2 v}\right)$,

when n is even

$$\cos n\theta = \cos^n \theta \prod_{p=1}^{p=\frac{n}{2}} \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{(2p-1)\pi}{2n}}\right) \tag{17}$$

$$\frac{\sin n\theta}{\sin \theta} = n \cos^{n-1}\theta \prod_{p=1}^{p=\frac{n-2}{2}} \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{p\pi}{n}}\right) \tag{18}$$

Serret also finds the corresponding formulae for the case in which n is odd.

§ 13. Factors of $\frac{\sinh nu}{\sinh u}$ and $\cosh nu$.

Corresponding to most of the factor forms of §§ 10, 11, there are factor forms for $\frac{\sinh nu}{\sinh u}$ and $\cosh nu$.

These forms are most readily obtained by putting $\theta = iu$ and making the assumption—a big assumption—that the results of §§ 10, 11, proved only for real values of θ, are also true when θ is imaginary, and using the formulae $\cos iu = \cosh u$, $\sin iu = i \sinh u$. The justification of the results depends on independent proofs that make no such assumption as that referred to.

Some of the results for $\frac{\sinh nu}{\sinh u}$ are found below, and similar proofs can be found for the various forms for $\cosh nu$.

From the general factor theorem,

$$\cosh nv - \cos na = 2^{n-1} \prod_{p=0}^{p=n-1} \left[\cosh v - \cos\left(a + \frac{2p\pi}{n}\right)\right].$$

Put $v = 2u$ and $a = 0$.

Then $$\cosh 2nu - 1 = 2^{n-1} \prod_{p=0}^{p=n-1} \left(\cosh 2u - \cos \frac{2p\pi}{n}\right).$$

$$\therefore\ 2 \sinh^2 nu = 2^{2n-1} \prod_{p=0}^{p=n-1} \left(\sinh^2 u + \sin^2 \frac{p\pi}{n}\right),$$

$$\therefore\ \frac{\sinh^2 nu}{\sinh^2 u} = 2^{2n-2} \prod_{p=1}^{p=n-1} \left(\sinh^2 u + \sin^2 \frac{p\pi}{n}\right).$$

Let $u \to 0$, then in the limit

$$n^2 = 2^{2n-2} \prod_{p=1}^{p=n-1} \sin^2 \frac{p\pi}{n}.$$

$$\therefore \quad \frac{\sinh^2 nu}{n^2 \sinh^2 u} = \prod_{p=1}^{p=n-1} \left(\frac{\sinh^2 u}{\sin^2 \frac{p\pi}{n}} + 1 \right).$$

Now factors equidistant from the ends are equal, and all the factors are positive, and $\sinh nu$ and $\sinh u$ are both positive or both negative.

$$\therefore \textit{ when } n \textit{ is odd, } \quad \frac{\sinh nu}{n \sinh u} = \prod_{p=1}^{p=\frac{n-1}{2}} \left(1 + \frac{\sinh^2 u}{\sin^2 \frac{p\pi}{n}} \right),$$

$$\textit{when } n \textit{ is even, } \quad \frac{\sinh nu}{n \sinh u \cosh u} = \prod_{p=1}^{p=\frac{n-2}{2}} \left(1 + \frac{\sinh^2 u}{\sin^2 \frac{p\pi}{n}} \right).$$

EXERCISE XXII. a.

Mainly on bookwork.

1. Write out in full the proof of formula (2) of § 3.

2. Show that the special case given at the end of § 4 can be changed to the form of formula (2) of § 3.

3. Deduce the factors of $x^n - 1$ from ($6a$) in § 7.

4. Deduce the factors of $x^n + 1$ from ($6a$) in § 7.

5. Prove that $1 - \dfrac{\sin^2 u}{\sin^2 v} = \cos^2 u \left(1 - \dfrac{\tan^2 u}{\tan^2 v} \right)$.

6. Use the transformation of No. 5 to show that:

(i) When n is even, $\dfrac{\sin n\theta}{\sin \theta} = n \cos^{n-1} \theta \displaystyle\prod_{p=1}^{p=\frac{n-2}{2}} \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{p\pi}{n}} \right)$.

(ii) When n is odd, $\dfrac{\sin n\theta}{\sin \theta} = n \cos^{n-1} \theta \displaystyle\prod_{p=1}^{p=\frac{n-1}{2}} \left(1 - \frac{\tan^2 \theta}{\tan^2 \frac{p\pi}{n}} \right)$.

7. Use the transformation of No. 5 to show that:

(i) When n is even, $\cos n\theta = \cos^n \theta \prod_{p=1}^{p=\frac{n}{2}} \left(1 - \dfrac{\tan^2 \theta}{\tan^2 \frac{(2p-1)\pi}{2n}}\right)$.

(ii) When n is odd, $\cos n\theta = \cos^n \theta \prod_{p=1}^{p=\frac{n-1}{2}} \left(1 - \dfrac{\tan^2 \theta}{\tan^2 \frac{(2p-1)\pi}{2n}}\right)$.

8. Use the method of § 12 to express (i) $\cos n\theta$, (ii) $\dfrac{\sin n\theta}{\sin \theta}$ in factors for the cases in which n is odd.

9. Show how the factors of $\dfrac{\sin n\theta}{\sin \theta}$ can be obtained from the general factor theorem, or from formula (3), § 4.

10. Use the general factor theorem to show that:

(i) When n is odd,

$$\cosh nu = 2^{n-1} \cosh u \prod_{p=1}^{p=\frac{n-1}{2}} \left(\sinh^2 u + \sin^2 \frac{\overline{2p-1}\,\pi}{2n}\right)$$

$$= \cosh u \prod_{p=1}^{p=\frac{n-1}{2}} \left(1 + \frac{\sinh^2 u}{\sin^2 \frac{\overline{2p-1}\,\pi}{2n}}\right).$$

(ii) When n is even,

$$\cosh nu = 2^{n-1} \prod_{p=1}^{p=\frac{n}{2}} \left(\sinh^2 u + \sin^2 \frac{\overline{2p-1}\,\pi}{2n}\right) = \prod_{p=1}^{p=\frac{n}{2}} \left(1 + \frac{\sinh^2 u}{\sin^2 \frac{\overline{2p-1}\,\pi}{2n}}\right).$$

11. Deduce the results of § 13 and No. 10 above from the results of §§ 10, 11 by putting $\theta = iu$.

12. Prove that $1 + \dfrac{\sinh^2 u}{\sin^2 v} = \cosh^2 u \left(1 + \dfrac{\tanh^2 u}{\tan^2 v}\right)$.

13. Use the transformation of No. 12 to deduce, from the results of No. 10, that:

(i) When n is odd, $\cosh nu = \cosh^n u \prod_{p=1}^{p=\frac{n-1}{2}} \left[1 + \dfrac{\tanh^2 u}{\tan^2 \frac{(2p-1)\pi}{2n}}\right]$.

(ii) When n is even, $\cosh nu = \cosh^n u \prod_{p=1}^{p=\frac{n}{2}} \left[1 + \dfrac{\tanh^2 u}{\tan^2 \frac{(2p-1)\pi}{2n}}\right]$.

14. Use the transformation of No. 12 to deduce, from the results of § 13, that:

(i) When n is odd, $$\frac{\sinh nu}{n \sinh u} = \cosh^{n-1} u \prod_{p=1}^{p=\frac{n-1}{2}} \left(1 + \frac{\tanh^2 u}{\tan^2 \frac{p\pi}{n}}\right).$$

(ii) When n is even, $$\frac{\sinh nu}{n \sinh u} = \cosh^{n-1} u \prod_{p=1}^{p=\frac{n-2}{2}} \left(1 + \frac{\tanh^2 u}{\tan^2 \frac{p\pi}{n}}\right).$$

§ 14. When given a factor form for simplification, the first thing to consider is what is the range of angles.

(i) If the range is 2π, it probably depends on the general factor theorem (§ 9).

(ii) If the range is π, it may come from the general factor theorem by taking together factors equidistant from the ends of the product, or by changing into half-angles $\left(\text{using } 1 + \cos \alpha = 2 \cos^2 \frac{\alpha}{2} \text{ perhaps}\right)$, or it may depend on some factor form that has been derived in one of these ways, *e.g.* the result of Example ii below.

(iii) If the range is $\frac{\pi}{2}$, it probably comes from a factor result of the type (ii) by taking together factors equidistant from the ends or by going into half-angles.

Example i. *Simplify*

$$2^{n-1} \sin \theta \sin\left(\theta + \frac{2\pi}{n}\right) \sin\left(\theta + 2\frac{2\pi}{n}\right) \dots \sin\left(\theta + \overline{n-1}\,\frac{2\pi}{n}\right).$$

The range of angles is 2π, so write

$$\sin\left(\theta + \frac{2p\pi}{n}\right) = \cos\frac{\pi}{2} - \cos\left(\frac{\pi}{2} + \theta + \frac{2p\pi}{n}\right).$$

$$\therefore \quad \text{the given expression} = 2^{n-1} \prod_{p=0}^{p=n-1} \left\{\cos\frac{\pi}{2} - \cos\left(\frac{\pi}{2} + \theta + \frac{2p\pi}{n}\right)\right\}$$

$$= \cos\frac{n\pi}{2} - \cos\left(n\frac{\pi}{2} + n\theta + 2p\pi\right)$$

$$= \cos\frac{n\pi}{2} - \cos\left(n\frac{\pi}{2} + n\theta\right).$$

Example ii. *Simplify*

$$2^{n-1} \sin\phi \sin\left(\phi + \frac{\pi}{n}\right) \sin\left(\phi + \frac{2\pi}{n}\right) \dots \sin\left(\phi + \frac{n-1}{n}\pi\right).$$

The range of angles is π, so write

$$2\sin^2\left(\phi+\frac{p\pi}{n}\right)=1-\cos\left(2\phi+\frac{2p\pi}{n}\right)=\cos 0-\cos\left(2\phi+\frac{2p\pi}{n}\right).$$

From § 4, $\cos n\theta-\cos n\alpha=2^{n-1}\prod_{p=0}^{p=n-1}\left\{\cos\theta-\cos\left(\alpha+\frac{2p\pi}{n}\right)\right\}$;

put $\theta=0$ and $\alpha=2\phi$, then

$$2\sin^2 n\phi=2^{n-1}\prod_{p=0}^{p=n-1}\left\{2\sin^2\left(\phi+\frac{p\pi}{n}\right)\right\}.$$

$$\therefore\quad \sin^2 n\phi=2^{2(n-1)}\prod_{p=0}^{p=n-1}\left\{\sin^2\left(\phi+\frac{p\pi}{n}\right)\right\},$$

$$\therefore\quad 2^{n-1}\sin\phi\sin\left(\phi+\frac{\pi}{n}\right)\sin\left(\phi+\frac{2\pi}{n}\right)\dots\sin\left(\phi+\frac{n-1}{n}\pi\right)=\pm\sin n\phi.$$

We have now to consider the ambiguous sign.

If ϕ lies between 0 and $\frac{\pi}{n}$, $\sin n\phi$ is +ve and all the factors are +ve.

If ϕ lies between $\frac{\pi}{n}$ and $\frac{2\pi}{n}$, $\sin n\phi$ is −ve and the last factor only is −ve.

If ϕ lies between $\frac{2\pi}{n}$ and $\frac{3\pi}{n}$, $\sin n\phi$ is +ve and the last two factors only are +ve.

etc. etc.

Therefore in all cases

$$\mathbf{2^{n-1}\sin\phi\sin\left(\phi+\frac{\pi}{n}\right)\sin\left(\phi+\frac{2\pi}{n}\right)\cdots\sin\left(\phi+\frac{n-1}{n}\pi\right)=\sin n\phi.}$$

.........(19)

Example iii. *To make various deductions from the result of Example ii.*

By changing ϕ into $\phi+\frac{\pi}{2n}$, we get

$$2^{n-1}\sin\left(\phi+\frac{\pi}{2n}\right)\sin\left(\phi+\frac{3\pi}{2n}\right)\dots\sin\left(\phi+\frac{2n-1}{2n}\pi\right)=\cos n\phi.$$

Again, as $\phi\to 0$, $\frac{\sin n\phi}{\sin\phi}\to n$.

$$\therefore\quad 2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\dots\sin\frac{n-1}{n}\pi=n,$$

but factors equidistant from the ends are equal and all the factors are positive.

$$\therefore\quad 2^{\frac{n-1}{2}}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\dots=\sqrt{n},$$

the last factor being $\sin\frac{n-1}{2n}\pi$ when n is odd, and $\sin\frac{n-2}{2n}\pi$ when n is even.

This last result was found by a different method near the end of § 10.

Again, put $\phi=\frac{\pi}{2}$,

$$2^{n-1}\cos\frac{\pi}{n}\cos\frac{2\pi}{n}\ldots\cos\frac{\overline{n-1}\,\pi}{n}=\sin\frac{n\pi}{2}=0 \text{ when } n \text{ is even,}$$

$$\text{or}=1 \text{ if } n \text{ is of the form } 4p+1,$$

$$\text{or}=-1 \text{ if } n \text{ is of the form } 4p-1.$$

If n is odd, by taking together factors equidistant from the ends and taking the square root

$$2^{\frac{n-1}{2}}\cos\frac{\pi}{n}\cos\frac{2\pi}{n}\ldots\cos\frac{\overline{n-1}\,\pi}{2n}=1,$$

since all the factors are positive.

Again, put $\phi=iu$ in (19), and we get

$$(-2i)^{n-1}\sinh u\sinh\left(u+i\frac{\pi}{n}\right)\sinh\left(u+i\frac{2\pi}{n}\right)\ldots\sinh\left(u+i\frac{\overline{n-1}\,\pi}{n}\right)=\sinh nu.$$

EXERCISE XXII. b.

1. Write out in linear factors:

(i) $\cos 5\alpha-\cos 5\beta$, (ii) $\cosh 4\alpha-\cos 4\beta$.

2. Express as the product of quadratic factors:

(i) $x^{2n}-2a^nx^n\cos n\theta+a^{2n}$, (iv) $x^8-2x^4\cos\frac{\pi}{4}+1$,

(ii) $x^4+\frac{1}{x^4}-2\cos 4\theta$, (v) $x^6+2x^3\cos\frac{\pi}{3}+1$.

(iii) $x^{12}-2x^6\cos 6\alpha+1$,

3. Write out (a) in linear factors, (b) in quadratic factors:

(i) $\cos 6\theta$, (ii) $\cos 7\theta$, (iii) $\frac{\sin 6\theta}{\sin\theta}$, (iv) $\frac{\sin 7\theta}{\sin\theta}$, (v) $\cosh 4u$, (vi) $\frac{\sinh 5u}{\sinh u}$.

4. Express as the product of real factors:

(i) x^6-x^3+1, (ii) $x^{12}-x^6+1$, (iii) $x^{10}+x^5+1$.

5. Express as the product of real factors:

(i) x^5-1, (ii) x^6-1, (iii) x^7+1, (iv) x^8+1.

6. Factorise:

(i) $\sin 3\alpha-\sin 3\beta$, (ii) $\sin 4\alpha-\sin 4\beta$.

7. Express as the product of real factors:

(i) $x^{2n}-a^{2n}$, (ii) $x^{2n+1}-a^{2n+1}$, (iii) $x^{2n}+a^{2n}$, (iv) $x^{2n+1}+a^{2n+1}$.

8. Prove that

$$\prod_{r=0}^{r=n-1}\left(\cos\phi-\cos\frac{2r\pi}{n}\right)+\prod_{r=0}^{r=n-1}\left\{1-\cos\left(\phi+\frac{2r\pi}{n}\right)\right\}=0.$$

9. Prove that

$$\frac{\cos n\theta - \cos na}{1 - \cos na} = \prod_{r=1}^{r=n} \left\{ 1 - \frac{\sin^2 \dfrac{\theta}{2}}{\sin^2 \left(\dfrac{r\pi}{n} + \dfrac{a}{2}\right)} \right\}$$

10. Prove that

$$\tan^2 \frac{n\phi}{2} = (-1)^{n} \prod_{r=0}^{r=n-1} \left[\frac{\cos \phi - \cos \dfrac{2r\pi}{n}}{\cos \phi - \cos \dfrac{(2r-1)\pi}{n}} \right]$$

11. Prove that

$$\cos a = 2^{n-1} \prod_{r=0}^{r=n-1} \sin \frac{2a + \overline{2r+1}\,\pi}{2n}.$$

12. (i) Prove that

$$2^{n-1} \cos \theta \cos \left(\theta + \frac{2\pi}{n}\right) \cos \left(\theta + \frac{4\pi}{n}\right) \ldots \cos \left(\theta + \frac{2\overline{n-1}\,\pi}{n}\right)$$
$$= (-1)^n \left(\cos \frac{n\pi}{2} - \cos n\theta\right).$$

(ii) Find the value of the product when n is odd.

(iii) Consider the special case in which θ is zero and n is even.

(iv) Put $n = 2p$ and deduce that, when p is odd,

$$2^{p-1} \cos \frac{\pi}{p} \cos \frac{2\pi}{p} \ldots \cos \frac{\overline{p-1}\,\pi}{p} = (-1)^{\frac{p-1}{2}}.$$

(v) Obtain from (iv) a factor form of sines by putting

$$\cos a = \sin \left(\frac{\pi}{2} - a\right).$$

13. What is the result of putting $n\theta = \frac{\pi}{2}$ in No. 12 (i)?

14. Prove that

$$2^{2n-1} \cos \frac{\pi}{n} \cos \frac{2\pi}{n} \cos \frac{3\pi}{n} \ldots \cos \frac{\overline{2n-1}\,\pi}{n} = (-1)^n - 1.$$

15. Prove that

$$\sin (2n+1)\theta = 2^{2n} \sin \theta \prod_{r=1}^{r=n} \left(\sin^2 \frac{r\pi}{2n+1} - \sin^2 \theta\right).$$

16. Prove that

$$2^{n-1} \cos \theta \cos \left(\theta + \frac{\pi}{n}\right) \cos \left(\theta + \frac{2\pi}{n}\right) \ldots \cos \left(\theta + \frac{\overline{n-1}\,\pi}{n}\right)$$

is equal to $(-1)^{\frac{n-1}{2}} \cos n\theta$ if n is odd, or $(-1)^{\frac{n}{2}} \sin n\theta$ if n is even.

17. (i) Prove that

$$2^{n-1}\cos\frac{\pi}{2n}\cos\frac{3\pi}{2n}\cos\frac{5\pi}{2n}\dots\cos\frac{\overline{2n-1}\,\pi}{2n}=\cos\frac{n\pi}{2}.$$

(ii) Obtain new forms for this equation by taking together factors equidistant from the ends.

(iii) Use the results of (ii) to find factor forms of sines by putting

$$\cos\alpha=\sin\left(\frac{\pi}{2}-\alpha\right).$$

18. (i) Use the fact that

$$\sin\alpha=2\sin\frac{\alpha}{2}\sin\frac{\alpha+\pi}{2}$$

to prove that, when n is a power of 2,

$$\sin\theta=2^{n-1}\sin\frac{\theta}{n}\sin\frac{\theta+\pi}{n}\sin\frac{\theta+2\pi}{n}\dots\sin\frac{\theta+\overline{n-1}\,\pi}{n}.$$

(ii) By taking together factors equidistant from the ends, show that

$$\sin\theta=2^{n-1}\sin\frac{\theta}{n}\cos\frac{\theta}{n}\left(\sin^2\frac{\pi}{n}-\sin^2\frac{\theta}{n}\right)\left(\sin^2\frac{2\pi}{n}-\sin^2\frac{\theta}{n}\right)\dots$$

$$\dots\left(\sin^2\frac{\overline{n/2-1}\,\pi}{n}-\sin^2\frac{\theta}{n}\right).$$

§ 15. Applications of Factors.

Example iv. *Prove that*

$$n\cot n\phi=\cot\phi+\cot\left(\phi+\frac{\pi}{n}\right)+\cot\left(\phi+\frac{2\pi}{n}\right)+\dots+\cot\left(\phi+\frac{\overline{n-1}\,\pi}{n}\right).$$

The range of angles is π.

In Example ii, p. 355, it is proved that

$$\sin n\phi=2^{n-1}\sin\phi\sin\left(\phi+\frac{\pi}{n}\right)\sin\left(\phi+\frac{2\pi}{n}\right)\dots\sin\left(\phi+\frac{\overline{n-1}\,\pi}{n}\right).$$

Taking logs of each side of this equation, we get

$$\log\sin n\phi=\log 2^{n-1}+\sum_{p=0}^{p=n-1}\log\sin\left(\phi+\frac{p\pi}{n}\right).$$

Differentiate with respect to ϕ.

$$\therefore\ n\cot n\phi=\sum_{p=0}^{p=n-1}\cot\left(\phi+\frac{p\pi}{n}\right).$$

Note. This method of deducing a series from a product is often very useful. See Chapter XXIII, § 8.

Example v. *Express* $\dfrac{x^5}{x^7+1}$ *in real partial fractions.*

If a is one of the roots of the equation $x^7+1=0$, then the partial fraction which has $(x-a)$ for its denominator is† $\dfrac{\frac{a^5}{7a^6}}{x-a}=\dfrac{\frac{1}{7a}}{x-a}$. Hence the two partial fractions corresponding to the conjugate roots $\left(1, \frac{p\pi}{7}\right)$, $\left(1, -\frac{p\pi}{7}\right)$ give

$$\frac{1}{7\left(1, \frac{p\pi}{7}\right)}\,\frac{1}{x-\left(1, \frac{p\pi}{7}\right)}+\frac{1}{7\left(1, -\frac{p\pi}{7}\right)}\,\frac{1}{x-\left(1, -\frac{p\pi}{7}\right)}$$

$$=\frac{1}{7}\,\frac{\left(1, -\frac{p\pi}{7}\right)\left\{x-\left(1, -\frac{p\pi}{7}\right)\right\}+\left(1, \frac{p\pi}{7}\right)\left\{x-\left(1, \frac{p\pi}{7}\right)\right\}}{x^2-2x\cos\frac{p\pi}{7}+1}$$

$$=\frac{1}{7}\,\frac{x\,.\,2\cos\frac{p\pi}{7}-2\cos\frac{2p\pi}{7}}{x^2-2x\cos\frac{p\pi}{7}+1}$$

$$=\frac{2}{7}\,\frac{x\cos\frac{p\pi}{7}-\cos\frac{2p\pi}{7}}{x^2-2x\cos\frac{p\pi}{7}+1}$$

and p has the values 1, 3, 5.

Also the partial fraction corresponding to the factor $(x+1)$ is $-\dfrac{1}{7}\,\dfrac{1}{(x+1)}$.

Hence

$$\frac{x^5}{x^7+1}=-\frac{1}{7(x+1)}+\frac{2}{7}\sum_{r=0}^{r=2}\frac{x\cos\frac{(2r+1)\pi}{7}-\cos\frac{2(2r+1)\pi}{7}}{x^2-2x\cos\frac{(2r+1)\pi}{7}+1}$$

EXERCISE XXII. c.

Prove the following:

1. $2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\dots\sin\frac{\overline{n-1}\,\pi}{n}=n.$

2. $2^{\frac{n-1}{2}}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\dots\sin\frac{\overline{n-1}\,\pi}{2n}=\sqrt{n}.$ $\quad n$ odd.

---

† If $\dfrac{f(x)}{\phi(x)}$ is to be put into partial fractions, and if $(x-a)$ is a factor of $\phi(x)$ which is not a repeated factor, the coefficient of the term $\dfrac{1}{x-a}$ is $\dfrac{f(a)}{\phi'(a)}$, where $\phi'(x)$ is the first derived function of $\phi(x)$.

Prove the following:

3. $2^{\frac{n-1}{2}} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \ldots \sin\frac{\overline{n-2}\,\pi}{2n} = \sqrt{n}$. n even.

4. $2^{n-1} \sin\frac{\pi}{2n} \sin\frac{3\pi}{2n} \ldots \sin\frac{\overline{2n-1}\,\pi}{2n} = 1$.

5. $2^{\frac{n-1}{2}} \sin\frac{\pi}{2n} \sin\frac{3\pi}{2n} \ldots \sin\frac{\overline{n-2}\,\pi}{2n} = 1$. n odd.

6. $2^{\frac{n-1}{2}} \cos\frac{\pi}{n} \cos\frac{2\pi}{n} \ldots \cos\frac{\overline{n-1}\,\pi}{2n} = 1$. n odd.

7. $2^{2n-1} \cos\frac{\pi}{n} \cos\frac{2\pi}{n} \ldots \cos\frac{\overline{2n-1}\,\pi}{n} = (-1)^n - 1$.

8. $2^{n-1} \cos\frac{\pi}{2n} \cos\frac{5\pi}{2n} \ldots \cos\frac{\overline{4n-3}\,\pi}{2n} = (-1)^n \cos\frac{n\pi}{2}$.

9. $2^{n-1} \cos\frac{\pi}{2n} \cos\frac{3\pi}{2n} \ldots \cos\frac{\overline{2n-1}\,\pi}{2n} = \cos\frac{n\pi}{2}$.

10. $2^{n-1} \cos\frac{\pi}{2n} \cos\frac{3\pi}{2n} \ldots \cos\frac{\overline{n-1}\,\pi}{2n} = 1$. n even.

11. $2^{\frac{n-1}{2}} \cos\frac{\pi}{2n} \cos\frac{3\pi}{2n} \ldots \cos\frac{\overline{n-2}\,\pi}{2n} = \sqrt{n}$. n odd.

12. $2^{n-\frac{1}{2}} \sin\frac{\pi}{4n} \sin\frac{3\pi}{4n} \ldots \sin\frac{\overline{2n-1}\,\pi}{4n} = 1$.

13. Prove that, when n is odd,

$$\tan\theta + \tan\left(\theta + \frac{2\pi}{n}\right) + \tan\left(\theta + \frac{4\pi}{n}\right) + \ldots + \tan\left(\theta + \frac{\overline{2n-1}\,\pi}{n}\right) = n\tan n\theta.$$

14. Prove that

$$\frac{n\sin n\phi}{\cos n\phi - \cos n\theta} = \sum_{r=0}^{r=n-1} \frac{\sin\phi}{\cos\phi - \cos(r\alpha + \theta)}, \text{ where } n\alpha = 2\pi.$$

15. Prove that

$$\frac{n\sin n\theta}{\cos n\phi - \cos n\theta} = \sum_{r=0}^{r=n-1} \frac{\sin\left(\theta + \frac{2r\pi}{n}\right)}{\cos\phi - \cos\left(\theta + \frac{2r\pi}{n}\right)}.$$

16. Express in real partial fractions $\frac{1}{1+x^5}$.

17. Resolve into the sum of n partial fractions:

$$\text{(i)}\ \frac{x^n - a^n \cos n\theta}{x^{2n} - 2x^n a^n \cos n\theta + a^{2n}}, \qquad \text{(ii)}\ \frac{nx^{n-1}(x^n - a^n \cos n\theta)}{x^{2n} - 2x^n a^n \cos n\theta + a^{2n}}.$$

18. Prove that

$$\frac{1}{\cos\frac{\pi}{n} - \cos\theta} + \frac{1}{\cos\frac{\pi}{n} - \cos\left(\theta + \frac{2\pi}{n}\right)} + \ldots + \frac{1}{\cos\frac{\pi}{n} - \cos\left(\theta + \frac{\overline{2n-1}\,\pi}{n}\right)} = 0.$$

19. Prove that

$$\sum_{r=0}^{r=n-1} \frac{1}{1 - \cos\left(\phi + \frac{2r\pi}{n}\right)} = \frac{n^2}{1 - \cos n\phi}.$$

20. If n is even, show that

$$(1+x)^n + (1-x)^n = 2\left(x^2 + \cot^2\frac{\pi}{2n}\right)\left(x^2 + \cot^2\frac{3\pi}{2n}\right)\ldots\left(x^2 + \cot^2\frac{\overline{n-1}\,\pi}{2n}\right).$$

21. If $a = \frac{\pi}{2n+1}$, prove that

$$(1+x)^{2n+1} - (1-x)^{2n+1} = 2(2n+1)x(1 + x^2\cot^2 a)(1 + x^2\cot^2 2a)\ldots(1 + x^2\cot^2 na).$$

22. If n is even, prove that

$$(1+x)^n - (1-x)^n = 2nx\left(x^2 + \tan^2\frac{\pi}{n}\right)\left(x^2 + \tan^2\frac{2\pi}{n}\right)\ldots\left(x^2 + \tan^2\frac{\overline{n-2}\,\pi}{2n}\right).$$

23. Prove that

$$x^{2n} + (x-1)^{2n} = 2\prod_{r=1}^{r=n}\left\{x^2 - x + \tfrac{1}{4}\operatorname{cosec}^2\frac{(2r-1)\pi}{4n}\right\}.$$

24. Prove that

$$\frac{1}{(1+x)^5 + (1-x)^5} = \frac{\mathsf{A}}{x^2 + \tan^2\frac{\pi}{10}} + \frac{\mathsf{B}}{x^2 + \tan^2\frac{3\pi}{10}},$$

and find the values of A and B.

25. Prove that, if $a = \frac{\pi}{n}$,

$$\frac{nx^{2n-1}}{x^{2n} - 1} = \frac{x}{x^2 - 1} + \sum_{r=1}^{r=n-1} \frac{x - \cos ra}{x^2 - 2x\cos ra + 1}.$$

26. Prove that

$$\cos n\theta + \sin n\theta = 2^{n-\frac{1}{2}} \prod_{p=0}^{p=n-1} \sin\left(\theta + \frac{\overline{4p+1}\,\pi}{4n}\right).$$

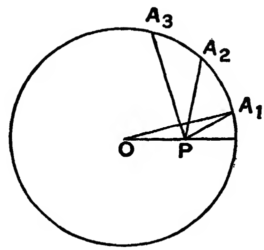
27. Prove that

$$\frac{1}{1 - \cos\frac{\pi}{2n}} + \frac{1}{1 - \cos\frac{3\pi}{2n}} + \ldots \text{ to } n \text{ terms} = n^2.$$

§ 16. De Moivre's Property of the Circle.

If $A_1, A_2, A_3, \ldots, A_n$ *be the successive vertices of a regular polygon* inscribed in a circle of radius a and centre* O, *and* P *a point distant x from* O, *such that* $\angle POA_1$ *is* θ, *then*

$$PA_1^2 . PA_2^2 . PA_3^2 . \ldots . PA_n^2 = x^{2n} - 2x^n a^n \cos n\theta + a^{2n}.$$



$$\angle POA_{r+1} = \theta + \frac{r . 2\pi}{n}.$$

$$\therefore\ PA^2_{r+1} = OP^2 + OA^2 - 2OP . OA \cos POA_{r+1}$$
$$= x^2 - 2xa \cos\left(\theta + \frac{r . 2\pi}{n}\right) + a^2.$$

$$\therefore\ PA_1^2 . PA_2^2 . PA_3^2 . \ldots . PA_n^2$$
$$= \prod_{r=0}^{r=n-1}\left[x^2 - 2xa\cos\left(\theta + \frac{r . 2\pi}{n}\right) + a^2\right]$$
$$= x^{2n} - 2x^n a^n \cos n\theta + a^{2n}, \text{ by § 7.}$$

Cotes' Properties of the Circle.

These are really particular cases of De Moivre's Property.

(i) If P is a point on OA_1, $\theta = 0$.

$$\therefore\ PA_1^2 . PA_2^2 . \ldots . PA_n^2 = x^{2n} - 2x^n a^n + a^{2n}.$$

$$\therefore\ PA_1 . PA_2 . \ldots . PA_n = x^n - a^n.$$

(ii) If P is on the bisector of $\angle A_nOA_1$, $\theta = \frac{\pi}{n}$.

$$\therefore\ PA_1^2 . PA_2^2 . \ldots . PA_n^2 = x^{2n} - 2x^n a^n \cos\pi + a^{2n}$$
$$= x^{2n} + 2x^n a^n + a^{2n}.$$

$$\therefore\ PA_1 . PA_2 . \ldots . PA_n = x^n + a^n.$$

EXERCISE XXII. d.

1. $A_1, A_2, \ldots, A_{2n}$ **are the vertices of a regular polygon of** $2n$ **sides inscribed in a circle of radius** a.

(i) **If O is the mid-point of the arc** $A_1 A_{2n}$, **prove that**

$$OA_1 . OA_2 . \ldots . OA_n = \sqrt{2}\, a^n.$$

(ii) **Prove that**

$$A_1 A_2 . A_1 A_3 . \ldots . A_1 A_n = a^{n-1}\sqrt{n}.$$

2. $A_1, A_2, \ldots, A_{2n+1}$ are the vertices of a regular polygon of $(2n+1)$ sides inscribed in a circle of radius a. If the diameter through A_{n+1} meets the circle again in P, prove that

$$PA_1 . PA_2 . \ldots . PA_n = a^n.$$

3. A regular polygon of $2n$ sides is inscribed in a circle of radius a. Prove that the product of the perpendiculars from the corners to a line through the centre and the mid-point of one side is $\dfrac{a^{2n}}{2^{2n-2}}$.

4. A regular polygon of n sides is inscribed in a circle of radius R and perpendiculars are let fall from its vertices on a chord. Show that their product is

$$\frac{R^n}{2^{n-1}}(\cos n\alpha + \cos n\beta),$$

where 2α is the angle subtended at the centre by the chord, and β is the angle it makes with any side of the polygon.

5. If a square $A_1 A_2 A_3 A_4$ and a regular pentagon $B_1 B_2 B_3 B_4 B_5$ be inscribed in a circle of radius r, show that the continued product of all the chords AB is numerically equal to $2r^{20} \sin 20\,\theta$, where 2θ is the angle subtended at the centre by any one of these chords.

6. If A, B, C, ... be the angular points of a regular polygon of n sides inscribed in a circle of radius a and centre O, and if P be any other point, show that the sum of the angles made by PA, PB, PC, ... with OP produced is

$$\tan^{-1} \frac{a^n \sin n\theta}{a^n \cos n\theta - x^n},$$

where $OP = x$ and $POA = \theta$.

CHAPTER XXIII

INFINITE PRODUCTS

§ 1. In Chapter XVIII it was explained that the theory of infinite series was not treated fully on the grounds that it is outside the scope of the present work and is best studied after some acquaintance with infinite series. For similar reasons we shall not deal with the fundamental difficulties of proceeding from the product of a finite number of factors to that of an infinite number of factors. The reader should consult such books as Hobson's *Trigonometry* and Chrystal's *Algebra*.

If Π_1^n denotes the product of n factors $u_1, u_2, \ldots, u_n$, where

$$u_1, u_2, \ldots, u_n$$

is a sequence of quantities formed according to some law, and if $\underset{n\to\infty}{\text{Lt}}\ \Pi_1^n$ tends to some finite limit Π, then Π is spoken of as the value of the **infinite product** and it is said to be a **convergent** product.

When we wish to express $f(x)$ in the form of an infinite product, we shall content ourselves with adopting one of the following courses:

(i) we shall find Π_1^n, the value of the product of the first n factors, and we shall prove that $\underset{n\to\infty}{\text{Lt}}\ \Pi_1^n = f(x)$;

or (ii) we shall show that $f(x) = \Pi_1^n \times \text{Q}$, where Q lies between 1 and a quantity which approaches 1 as n tends to infinity.

§ 2. To prove that

$\dfrac{\sin\theta}{\theta} = \cos\dfrac{\theta}{2}\cos\dfrac{\theta}{2^2}\cos\dfrac{\theta}{2^3}\ldots$ ad infin. (Euler's Product).

$$\begin{aligned}\sin\theta &= 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\\ &= 2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\sin\frac{\theta}{2^2}\\ &= 2^3\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\sin\frac{\theta}{2^3}\\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &= 2^n\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n}.\end{aligned}$$

$$\therefore \quad \Pi_1^n \equiv \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\cos\frac{\theta}{2^n}=\frac{\sin\theta}{2^n\sin\dfrac{\theta}{2^n}}.$$

$$\therefore \quad \underset{n\to\infty}{\mathrm{Lt}}\ \Pi_1^n = \underset{n\to\infty}{\mathrm{Lt}}\left(\frac{\sin\theta}{\theta}\,\frac{\dfrac{\theta}{2^n}}{\sin\dfrac{\theta}{2^n}}\right)$$

$$=\frac{\sin\theta}{\theta}\ \underset{n\to\infty}{\mathrm{Lt}}\ \frac{\dfrac{\theta}{2^n}}{\sin\dfrac{\theta}{2^n}}$$

$$=\frac{\sin\theta}{\theta}. \qquad \text{(Chap. IX, § 4, p. 139.)}$$

$$\therefore \quad \frac{\sin\theta}{\theta}=\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\ \textit{ad infin.}$$

§ 3. To show that when ϕ is real,

$$\sin\phi=\phi\left(1-\frac{\phi^2}{\pi^2}\right)\left(1-\frac{\phi^2}{2^2\pi^2}\right)\left(1-\frac{\phi^2}{3^2\pi^2}\right)\ldots \textbf{ ad infin.}$$

$$=\phi\prod_{p=1}^{p=\infty}\left(1-\frac{\phi^2}{p^2\pi^2}\right).$$

From Chap. XXII, § 10, p. 348, when n is even

$$\frac{\sin n\theta}{n\sin\theta\cos\theta}=\left(1-\frac{\sin^2\theta}{\sin^2\dfrac{\pi}{n}}\right)\left(1-\frac{\sin^2\theta}{\sin^2\dfrac{2\pi}{n}}\right)\left(1-\frac{\sin^2\theta}{\sin^2\dfrac{3\pi}{n}}\right)\ldots$$

$$\ldots\left(1-\frac{\sin^2\theta}{\sin^2\dfrac{n-2}{2}\dfrac{\pi}{n}}\right).$$

Let $n\theta=\phi$, and let ϕ remain finite, but let $n\to\infty$, so that $\theta=\dfrac{\phi}{n}\to 0$.

Then $$n \sin\theta\cos\theta = n\theta\frac{\sin\theta}{\theta}\cos\frac{\phi}{n} = \phi\frac{\sin\theta}{\theta}\cos\frac{\phi}{n}.$$

$$\therefore\ \sin\phi = \phi\frac{\sin\theta}{\theta}\cos\frac{\phi}{n}\left(1-\frac{\sin^2\frac{\phi}{n}}{\sin^2\frac{\pi}{n}}\right)\left(1-\frac{\sin^2\frac{\phi}{n}}{\sin^2\frac{2\pi}{n}}\right)$$

$$\times\left(1-\frac{\sin^2\frac{\phi}{n}}{\sin^2\frac{3\pi}{n}}\right)\cdots\left(1-\frac{\sin^2\frac{\phi}{n}}{\sin^2\frac{r\pi}{n}}\right)Q,$$

where $$Q = \left(1-\frac{\sin^2\frac{\phi}{n}}{\sin^2\frac{\overline{r+1}\,\pi}{n}}\right)\left(1-\frac{\sin^2\frac{\phi}{n}}{\sin^2\frac{\overline{r+2}\,\pi}{n}}\right)\cdots$$

$$\cdots\left(1-\frac{\sin^2\frac{\phi}{n}}{\sin^2\frac{n-2}{2}\frac{\pi}{n}}\right)$$

Now, as $n\to\infty$, $\dfrac{\sin\theta}{\theta}\to 1$, $\cos\dfrac{\phi}{n}\to 1$

and $$1-\frac{\sin^2\frac{\phi}{n}}{\sin^2\frac{p\pi}{n}} = 1-\frac{\sin^2\frac{\phi}{n}}{\left(\frac{\phi}{n}\right)^2}\times\frac{\left(\frac{p\pi}{n}\right)^2}{\sin^2\frac{p\pi}{n}}\times\frac{\phi^2}{p^2\pi^2}\to 1-\frac{\phi^2}{p^2\pi^2}.$$

$$\therefore\ \sin\phi = \phi\left(1-\frac{\phi^2}{\pi^2}\right)\left(1-\frac{\phi^2}{2^2\pi^2}\right)\left(1-\frac{\phi^2}{3^2\pi^2}\right)\cdots\left(1-\frac{\phi^2}{r^2\pi^2}\right)Q',$$

where Q is the value of Q as $n\to\infty$

To complete the investigation it is necessary to prove that Q' lies between 1 and $1-\epsilon$, where ϵ is a positive quantity which approaches 0 as r gets larger and larger.

Since we are at liberty to choose r as we please, we may choose it so that

$$\left(\frac{\sin\frac{\phi}{n}}{\sin\frac{p\pi}{n}}\right)^2 < 1$$

when $p = r+1$ or any number $> r+1$; hence we see that each factor in Q is positive and less than 1.

$$\therefore \quad Q < 1.$$

The proof that $Q >$ a quantity which approaches 1 as $r \to \infty$ is more difficult. The reader should consult Hobson's *Trigonometry* for the completion of this proof. See also § 6.

Note we might equally well have started from the product for $\sin n\theta$ when n is odd, formula (12a), p. 348.

Note also the following form for the product:

$$\sin\phi = \phi\left(1 - \frac{4\phi^2}{2^2\pi^2}\right)\left(1 - \frac{4\phi^2}{4^2\pi^2}\right)\ldots = \phi\prod_{p=1}^{p=\infty}\left(1 - \frac{4\phi^2}{(2p)^2\pi^2}\right).$$

§ 4. By an investigation similar to that of § 3, from the formula (16a) or (16b), p. 349, for $\cos n\theta$, we can deduce that, when ϕ is real,

$$\boldsymbol{\cos\phi = \left(1 - \frac{4\phi^2}{\pi^2}\right)\left(1 - \frac{4\phi^2}{3^2\pi^2}\right)\left(1 - \frac{4\phi^2}{5^2\pi^2}\right)\ldots \text{ ad infin.}}$$

$$\boldsymbol{= \prod_{p=1}^{p=\infty}\left(1 - \frac{4\phi^2}{(2p-1)^2\pi^2}\right).}$$

See also § 6.

The infinite product for $\cos\phi$ can also be obtained by dividing the product for $\sin 2\phi$ by that for $2\sin\phi$.

§ 5. The infinite products for $\sin\phi$ and $\cos\phi$ have been proved for the case in which ϕ is real; they can also be proved to be true when ϕ is imaginary or complex. See Hobson's *Trigonometry.*

If we put $\phi = iu$ we get, from § 3,

$$\sin iu = iu\left(1 + \frac{u^2}{\pi^2}\right)\left(1 + \frac{u^2}{2^2\pi^2}\right)\left(1 + \frac{u^2}{3^2\pi^2}\right)\ldots \textit{ ad infin.}$$

Now $\sin iu = i \sinh u$.

$$\therefore\ \mathbf{\sinh u = u\left(1+\frac{u^2}{\pi^2}\right)\left(1+\frac{u^2}{2^2\pi^2}\right)\left(1+\frac{u^2}{3^2\pi^2}\right) \ldots\ ad\ infin.}$$

$$\mathbf{= u\prod_{p=1}^{p=\infty}\left(1+\frac{u^2}{p^2\pi^2}\right).}$$

Similarly, from § 4,

$$\mathbf{\cosh u = \left(1+\frac{4u^2}{\pi^2}\right)\left(1+\frac{4u^2}{3^2\pi^2}\right)\left(1+\frac{4u^2}{5^2\pi^2}\right) \ldots\ ad\ infin.}$$

$$\mathbf{= \prod_{p=1}^{p=\infty}\left[1+\frac{4u^2}{(2p-1)^2\pi^2}\right].}$$

These products can be derived from the results of Chap. XXII, § 13, and similar results.

§ 6. The infinite products for $\sin\phi$ and $\cos\phi$ can also be obtained from formulae of Ex. XXII. a, Nos. 6, 7. These formulae are not so easily obtained as those used in §§ 3, 4, but the investigation of the convergence is a little easier.

To show that, when ϕ is real,

$$\mathbf{\cos\phi = \left(1-\frac{4\phi^2}{\pi^2}\right)\left(1-\frac{4\phi^2}{3^2\pi^2}\right)\left(1-\frac{4\phi^2}{5^2\pi^2}\right) \ldots\ ad\ infin.}$$

From Ex. XXII. a, No. 7, when n is even,

$$\cos n\theta = \cos^n\theta\left(1-\frac{\tan^2\theta}{\tan^2\frac{\pi}{2n}}\right)\left(1-\frac{\tan^2\theta}{\tan^2\frac{3\pi}{2n}}\right)\left(1-\frac{\tan^2\theta}{\tan^2\frac{5\pi}{2n}}\right)\cdots$$

$$\cdots\left(1-\frac{\tan^2\theta}{\tan^2\frac{(n-1)\pi}{2n}}\right).$$

Let $n\theta=\phi$, and let ϕ remain finite, but let $n\to\infty$, so that $\theta=\frac{\phi}{n}\to 0$.

$$\therefore\ \cos\phi = \cos^n\frac{\phi}{n}\left(1-\frac{\tan^2\frac{\phi}{n}}{\tan^2\frac{\pi}{2n}}\right)\left(1-\frac{\tan^2\frac{\phi}{n}}{\tan^2\frac{3\pi}{2n}}\right).\left(1-\frac{\tan^2\frac{\phi}{n}}{\tan^2\frac{5\pi}{2n}}\right)\cdots$$

$$\cdots\left(1-\frac{\tan^2\frac{\phi}{n}}{\tan^2\frac{(2r-1)\pi}{2n}}\right)Q,$$

$$\text{where } Q = \left(1 - \frac{\tan^2 \dfrac{\phi}{n}}{\tan^2 \dfrac{(2r+1)\pi}{2n}}\right)\left(1 - \frac{\tan^2 \dfrac{\phi}{n}}{\tan^2 \dfrac{(2r+3)\pi}{2n}}\right)\cdots$$

$$\cdots\left(1 - \frac{\tan^2 \dfrac{\phi}{n}}{\tan^2 \dfrac{(n-1)\pi}{2n}}\right).$$

Now, as $n \to \infty$, $\cos^n \dfrac{\phi}{n} \to 1$,

$$\text{and } 1 - \frac{\tan^2 \dfrac{\phi}{n}}{\tan^2 \dfrac{p\pi}{2n}} = 1 - \left(\frac{\tan \dfrac{\phi}{n}}{\dfrac{\phi}{n}}\right)^2 \times \left(\frac{\dfrac{p\pi}{2n}}{\tan \dfrac{p\pi}{2n}}\right)^2 \times \frac{4\phi^2}{p^2\pi^2} \to 1 - \frac{4\phi^2}{p^2\pi^2}.$$

$$\therefore \cos\phi = \left(1 - \frac{4\phi^2}{\pi^2}\right)\left(1 - \frac{4\phi^2}{3^2\pi^2}\right)\left(1 - \frac{4\phi^2}{5^2\pi^2}\right)\cdots\left(1 - \frac{4\phi^2}{(2r-1)^2\pi^2}\right) Q',$$

where Q' is the value of Q as $n \to \infty$.

We shall now prove that Q' lies between 1 and $1-\epsilon$, where ϵ is a positive quantity which approaches 0 as r gets larger and larger.

Since we are at liberty to choose r as we please, we may choose it so that

$$\left(\frac{\tan \dfrac{\phi}{n}}{\tan \dfrac{p\pi}{2n}}\right)^2 < 1,$$

where $p = 2r-1$ or any number $> 2r-1$; hence we see that each factor of Q is positive and <1.

$$\therefore Q \text{ is } < 1.$$

Again

$$Q = \left(1 - \frac{\tan^2\frac{\phi}{n}}{\tan^2\frac{(2r+1)\pi}{2n}}\right)\left(1 - \frac{\tan^2\frac{\phi}{n}}{\tan^2\frac{(2r+3)\pi}{2n}}\right)\cdots$$

$$\cdots\left(1 - \frac{\tan^2\frac{\phi}{n}}{\tan^2\frac{(n-1)\pi}{2n}}\right)$$

$$> 1 - \tan^2\frac{\phi}{n}\left\{\frac{1}{\tan^2\frac{(2r+1)\pi}{2n}} + \frac{1}{\tan^2\frac{(2r+3)\pi}{2n}} + \dots + \frac{1}{\tan^2\frac{(n-1)\pi}{2n}}\right\}^{\dagger}$$

$$> 1 - \tan^2\frac{\phi}{n}\left\{\frac{4n^2}{(2r+1)^2\pi^2} + \frac{4n^2}{(2r+3)^2\pi^2} + \dots + \frac{4n^2}{(n-1)^2\pi^2}\right\},$$

since $\tan\frac{p\pi}{2n} > \frac{p\pi}{2n}$, as $\frac{p\pi}{2n}$ is an acute angle for all the values of p in the range.

$$\therefore\quad Q > 1 - \frac{4n^2}{\pi^2}\tan^2\frac{\phi}{n}\left\{\frac{1}{(2r+1)^2} + \frac{1}{(2r+3)^2} + \dots + \frac{1}{(n-1)^2}\right\}$$

$$> 1 - \frac{4n^2}{\pi^2}\tan^2\frac{\phi}{n}\left\{\frac{1}{2r(2r+2)} + \frac{1}{(2r+2)(2r+4)} + \dots + \frac{1}{(n-2)n}\right\},$$

$$\text{i.e.} > 1 - \frac{4n^2}{\pi^2}\tan^2\frac{\phi}{n}\,\frac{1}{2}\left\{\frac{1}{2r} - \frac{1}{2r+2} + \frac{1}{2r+2} - \frac{1}{2r+4} + \dots + \frac{1}{n-2} - \frac{1}{n}\right\},$$

$$\text{i.e.} > 1 - \frac{2n^2}{\pi^2}\tan^2\frac{\phi}{n}\left\{\frac{1}{2r} - \frac{1}{n}\right\}$$

$$> 1 - \frac{2\phi^2}{\pi^2}\left(\frac{\tan\frac{\phi}{n}}{\frac{\phi}{n}}\right)^2\frac{1}{2r}.$$

† If $u_1, u_2, u_3, \dots$ are each positive and each <1,

$$(1-u_1)(1-u_2)(1-u_3)\dots > 1 - (u_1+u_2+u_3+\dots).$$

$$\therefore \quad Q' > 1 - \frac{2\phi^2}{\pi^2} \times \frac{1}{2r}.$$

$\therefore \quad Q' > 1 - \epsilon$, where $\epsilon \to 0$, as r gets larger and larger.

$$\therefore \quad \cos\phi = \left(1 - \frac{4\phi^2}{\pi^2}\right)\left(1 - \frac{4\phi^2}{3^2\pi^2}\right)\left(1 - \frac{4\phi^2}{5^2\pi^2}\right) \dots \textit{ad infin.}$$

The proof of $\sin\phi$ follows the same lines, starting from the formula of Ex. XXII. a, No. 6; the end part of the investigation is as follows:

$$Q > 1 - \frac{n^2}{\pi^2}\tan^2\frac{\phi}{n}\left\{\frac{1}{(r+1)^2} + \frac{1}{(r+2)^2} + \dots + \frac{1}{\left(\frac{n-2}{2}\right)^2}\right\}$$

$$> 1 - \frac{n^2}{\pi^2}\tan^2\frac{\phi}{n}\left\{\frac{1}{r(r+1)} + \frac{1}{(r+1)(r+2)} + \dots + \frac{1}{\left(\frac{n-4}{2}\right)\left(\frac{n-2}{2}\right)}\right\},$$

$$\textit{i.e.} > 1 - \frac{n^2}{\pi^2}\tan^2\frac{\phi}{n}\left\{\frac{1}{r} - \frac{1}{\frac{n-2}{2}}\right\}$$

$$> 1 - \frac{\phi^2}{\pi^2}\left(\frac{\tan\frac{\phi}{n}}{\frac{\phi}{n}}\right)^2\frac{1}{r}, \text{ etc.}$$

EXERCISE XXIII. a.

Mainly on bookwork.

1. Find the infinite product for $\sin\phi$ from formula (12*a*), p. 348, for $\sin n\theta$ when n is odd (see § 3).

2. Find the infinite product for $\cos\phi$ from formula (16*b*), p. 349, for $\cos n\theta$ when n is even (see § 4).

3. Deduce the infinite product of $\cos\phi$ from those of $\sin 2\phi$ and $\sin\phi$ (see § 4).

4. Find the infinite product for $\sin\phi$ from the formula of Ex. XXII. a, No. 6, when n is even (see end of § 6).

§ 7. **Example i.** *Prove that*

$$\cos 2\phi+\cos 2\theta=2\cos^2\theta\prod_{p=1}^{p=\infty}\left[\left\{1-\frac{4\phi^2}{(\overline{2p-1}\,\pi-2\theta)^2}\right\}\left\{1-\frac{4\phi^2}{(\overline{2p-1}\,\pi+2\theta)^2}\right\}\right].$$

$\cos 2\phi+\cos 2\theta=2\cos(\theta+\phi)\cos(\theta-\phi)$

$$=2\prod_{p=1}^{p=\infty}\left[1-\frac{4(\theta+\phi)^2}{(2p-1)^2\pi^2}\right]\times\prod_{p=1}^{p=\infty}\left[1-\frac{4(\theta-\phi)^2}{(2p-1)^2\pi^2}\right]$$

$$=2\prod_{p=1}^{p=\infty}\left[\frac{(\overline{2p-1}\,\pi-2\theta-2\phi)(\overline{2p-1}\,\pi+2\theta+2\phi)(\overline{2p-1}\,\pi-2\theta+2\phi)(\overline{2p-1}\,\pi+2\theta-2\phi)}{(2p-1)^4\pi^4}\right]$$

$$=2\prod_{p=1}^{p=\infty}\frac{[(\overline{2p-1}\,\pi-2\theta)^2-4\phi^2][(\overline{2p-1}\,\pi+2\theta)^2-4\phi^2]}{(2p-1)^4\pi^4}.$$

Put $\phi=0$.

Then $$1+\cos 2\theta=2\prod_{p=1}^{p=\infty}\frac{(\overline{2p-1}\,\pi-2\theta)^2(\overline{2p-1}\,\pi+2\theta)^2}{(2p-1)^4\pi^4}.$$

$$\therefore\ \frac{\cos 2\phi+\cos 2\theta}{1+\cos 2\theta}=\prod_{p=1}^{p=\infty}\left[\left\{1-\frac{4\phi^2}{(\overline{2p-1}\,\pi-2\theta)^2}\right\}\left\{1-\frac{4\phi^2}{(\overline{2p-1}\,\pi+2\theta)^2}\right\}\right],$$

$$\therefore\ \cos 2\phi+\cos 2\theta=2\cos^2\theta\prod_{p=1}^{p=\infty}\left[\left\{1-\frac{4\phi^2}{(\overline{2p-1}\,\pi-2\theta)^2}\right\}\left\{1-\frac{4\phi^2}{(\overline{2p-1}\,\pi+2\theta)^2}\right\}\right].$$

EXERCISE XXIII. b.

1. Prove that the value of the infinite product

$$\left(1-\tan^2\frac{a}{2}\right)\left(1-\tan^2\frac{a}{4}\right)\left(1-\tan^2\frac{a}{8}\right)\ldots\ldots \text{ is } \frac{a}{\tan a}.$$

2. Prove that

$$(2\cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ \ldots\ (2\cos 2^{n-1}\theta-1)=\frac{2\cos 2^n\theta+1}{2\cos\theta+1}.$$

3. Prove that

$$\frac{\sin(\theta+\phi)}{\sin\theta}=\left(1+\frac{\phi}{\theta}\right)\prod_{p=1}^{p=\infty}\left(1+\frac{\phi}{\theta-p\pi}\right)\left(1+\frac{\phi}{\theta+p\pi}\right).$$

Write down the corresponding result for $\dfrac{\sin(\theta-\phi)}{\sin\theta}$.

4. Prove that

$$\frac{\cos(\theta+\phi)}{\cos\theta}=\prod_{p=1}^{p=\infty}\left(1+\frac{2\phi}{2\theta-\overline{2p-1}\,\pi}\right)\left(1+\frac{2\phi}{2\theta+\overline{2p-1}\,\pi}\right).$$

Write down the corresponding result for $\dfrac{\cos(\theta-\phi)}{\cos\theta}$.

5. Express $\cos 2\phi-\cos 2\theta$ as an infinite product (use the method of Example i) and hence show that

$$\cos 2\phi-\cos 2\theta=2\sin^2\theta\left(1-\frac{\phi^2}{\theta^2}\right)\prod_{p=1}^{p=\infty}\left[\left\{1-\frac{\phi^2}{(p\pi-\theta)^2}\right\}\left\{1-\frac{\phi^2}{(p\pi+\theta)^2}\right\}\right].$$

6. Show that

$$\frac{\sin\theta+\sin\phi}{\sin\theta}=\left(1+\frac{\phi}{\theta}\right)\left(1-\frac{\phi}{\theta-\pi}\right)\left(1-\frac{\phi}{\theta+\pi}\right)\left(1+\frac{\phi}{\theta-2\pi}\right)$$
$$\times\left(1+\frac{\phi}{\theta+2\pi}\right)\left(1-\frac{\phi}{\theta-3\pi}\right)\left(1-\frac{\phi}{\theta+3\pi}\right)\dots \textit{ad infin.}$$

Write down the corresponding results for $\dfrac{\sin\theta-\sin\phi}{\sin\theta}$.

7. Show that:

(i) $$\frac{\cosh u+\cosh v}{1+\cosh v}=\prod_{p=1}^{p=\infty}\left(1+\frac{u^2+2uv}{(2p-1)^2\pi^2+v^2}\right)\left(1+\frac{u^2-2uv}{(2p-1)^2\pi^2+v^2}\right),$$

(ii) $$\frac{\cosh u-\cosh v}{1-\cosh v}=\left(1-\frac{u^2}{v^2}\right)\prod_{p=1}^{p=\infty}\left(1+\frac{u^2+2uv}{(2p)^2\pi^2+v^2}\right)\left(1+\frac{u^2-2uv}{(2p)^2\pi^2+v^2}\right),$$

(iii) $$\frac{\sinh u+\sinh v}{\sinh v}=\left(1+\frac{u}{v}\right)\prod_{p=1}^{p=\infty}\left(1+\frac{u^2+(-1)^p\,2uv}{p^2\pi^2+v^2}\right).$$

8. Express $\cosh u+\cos\theta$ as an infinite product. [Put $\cosh u=\cos iu$, and proceed as in Example i.]

Hence show that

$$\cosh u+\cos\theta=2\cos^2\frac{\theta}{2}\prod_{p=1}^{p=\infty}\left\{1+\frac{u^2}{(\overline{2p-1}\,\pi-\theta)^2}\right\}\left\{1+\frac{u^2}{(\overline{2p-1}\,\pi+\theta)^2}\right\}.$$

9. Show that

$$\cosh u-\cos\theta=2\sin^2\frac{\theta}{2}\left(1+\frac{u^2}{\theta^2}\right)\prod_{p=1}^{p=\infty}\left\{1+\frac{u^2}{(2p\pi-\theta)^2}\right\}\left\{1+\frac{u^2}{(2p\pi+\theta)^2}\right\}.$$

10. Show that:

(i) $$\cosh u+\cos u=2\prod_{p=1}^{p=\infty}\left\{1+\frac{4u^4}{(2p-1)^4\pi^4}\right\},$$

(ii) $$\cosh u-\cos u=u^2\cdot\prod_{p=1}^{p=\infty}\left\{1+\frac{4u^4}{(2p)^4\pi^4}\right\}.$$

11. Prove that

$$\sqrt{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)=\frac{1}{2}(\pi+2x)\left\{1-\frac{(\pi+2x)^2}{4^2\pi^2}\right\}\left\{1-\frac{(\pi+2x)^2}{8^2\pi^2}\right\}\dots.$$

12. Prove that

$$\prod_{p=1}^{p=\infty}\left\{1-\frac{4a^2}{(a+n\pi)^2}\right\}\left\{1-\frac{4a^2}{(a-n\pi)^2}\right\}=\frac{1}{3}+\frac{2}{3}\cos 2a.$$

13. Prove that

$$\prod_{p=1}^{p=\infty}\left(1+\frac{1}{p^2}\right)=\frac{\sinh\pi}{\pi}.$$

14. Assuming the infinite product for $\sin\theta$ and the series for $\sin\theta$ and $\cos\theta$ in ascending powers of θ, prove that

$$(1+x)\left(1-\frac{x}{3}\right)\left(1+\frac{x}{5}\right)\left(1-\frac{x}{7}\right)\dots=1+\frac{\pi}{4}\frac{x}{\underline{|1}}-\frac{\pi^2}{4^2}\frac{x^2}{\underline{|2}}-\frac{\pi^3}{4^3}\frac{x^3}{\underline{|3}}+\frac{\pi^4}{4^4}\frac{x^4}{\underline{|4}}+\frac{\pi^5}{4^5}\frac{x^5}{\underline{|5}}-\dots.$$

§ 8. Applications of Infinite Products.

Example ii. *Prove that, except when θ is a multiple of π,*

$$\cot\theta=\frac{1}{\theta}+\sum_{p=1}^{p=\infty}\left\{\frac{1}{\theta-p\pi}+\frac{1}{\theta+p\pi}\right\},$$

and deduce another expansion by differentiation.

$$\sin\theta=\theta\prod_{p=1}^{p=\infty}\left(1-\frac{\theta^2}{p^2\pi^2}\right).$$

$$\therefore\ \log\sin\theta=\log\theta+\sum_{p=1}^{p=\infty}\{\log(p\pi-\theta)+\log(p\pi+\theta)-\log p^2\pi^2\}.$$

Differentiate each side with respect to θ.

$$\therefore\ \cot\theta=\frac{1}{\theta}+\sum_{p=1}^{p=\infty}\left\{\frac{-1}{p\pi-\theta}+\frac{1}{p\pi+\theta}\right\}$$

$$=\frac{1}{\theta}+\sum_{p=1}^{p=\infty}\left\{\frac{1}{\theta-p\pi}+\frac{1}{\theta+p\pi}\right\}.$$

By differentiating again, we get

$$\operatorname{cosec}^2\theta=\frac{1}{\theta^2}+\sum_{p=1}^{p=\infty}\left\{\frac{1}{(\theta-p\pi)^2}+\frac{1}{(\theta+p\pi)^2}\right\}.$$

Example iii. *Prove that*

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\text{ ad infin.}=\frac{\pi^2}{6},$$

$$\textit{and}\quad\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots\text{ ad infin.}=\frac{\pi^4}{90}.$$

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\dots,$$

$$\text{also}\quad\sin\theta=\theta-\frac{\theta^3}{\underline{|3}}+\frac{\theta^5}{\underline{|5}}-\dots.$$

$$\therefore\ \left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\dots=1-\frac{\theta^2}{6}+\frac{\theta^4}{120}-\dots,$$

$$\therefore\ \log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\dots=\log\left\{1-\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\dots\right)\right\}.$$

Expanding each term and collecting in powers of θ,

$$-\frac{\theta^2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right)-\frac{1}{2}\frac{\theta^4}{\pi^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right)-\dots$$

$$=-\frac{\theta^2}{6}+\theta^4\left(\frac{1}{120}-\frac{1}{2}\cdot\frac{1}{36}\right)-\dots$$

$$=-\frac{\theta^2}{6}-\frac{\theta^4}{180}.$$

Equating coefficients of θ^2, and equating coefficients of θ^4, we get the required results.

Example iv. *Prove that*

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots \text{ ad infin. } (\textit{Wallis' Theorem.})$$

$$\frac{\sin\theta}{\theta}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots.$$

Put $\theta=\frac{\pi}{2}$, then $\frac{\theta}{\pi}=\frac{1}{2}$.

Hence
$$\frac{2}{\pi}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\ldots$$

$$=\frac{1.3}{2.2}\cdot\frac{3.5}{4.4}\cdot\frac{5.7}{6.6}\cdot\ldots.$$

$$\therefore\ \frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ \ldots.$$

Wallis' Theorem is also stated thus:
When n is very great

$$\sqrt{\tfrac{1}{2}\pi(2n+1)}=\frac{2.4.6.\ldots.2n}{1.3.5.\ldots.(2n-1)} \text{ approximately.}$$

For it is approximately true that, when n is large,

$$\frac{2}{\pi}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{(2n)^2}\right).$$

$$\therefore\ \frac{2}{\pi}=\frac{1.3^2.5^2.\ldots.(2n-1)^2(2n+1)}{2^2.4^2.6^2.\ldots.(2n)^2},$$

$$\therefore\ \sqrt{\tfrac{1}{2}\pi(2n+1)}=\frac{2.4.6.\ldots.2n}{1.3.5.\ldots.(2n-1)}.$$

EXERCISE XXIII. c.

APPLICATIONS OF INFINITE PRODUCTS.

1. Assuming the result of Ex. XXIII. b, No. 9, sum to infinity

$$\frac{1}{\phi^2}+\frac{1}{(2\pi-\phi)^2}+\frac{1}{(2\pi+\phi)^2}+\frac{1}{(4\pi-\phi)^2}+\frac{1}{(4\pi+\phi)^2}+\ldots.$$

2. Prove that $\quad\dfrac{1}{1+2^2}+\dfrac{1}{1+6^2}+\dfrac{1}{1+10^2}+\ldots=\dfrac{\pi}{8}\tanh\dfrac{\pi}{4}.$

3. Sum the series $\quad\dfrac{1}{1+4^2}+\dfrac{1}{1+8^2}+\dfrac{1}{1+12^2}+\ldots.$

4. Prove that $\quad\displaystyle\sum_{p=1}^{p=\infty}\frac{2}{1+p^2}=\pi\coth\pi-1.$

5. Prove that:

(i) $$\tan\theta = 8\theta\left\{\frac{1}{\pi^2-4\theta^2}+\frac{1}{3^2\pi^2-4\theta^2}+\frac{1}{5^2\pi^2-4\theta^2}+\dots\right\},$$

(ii) $$\tfrac{1}{4}\sec^2\theta = \frac{1}{(\pi-2\theta)^2}+\frac{1}{(\pi+2\theta)^2}+\frac{1}{(3\pi-2\theta)^2}+\frac{1}{(3\pi+2\theta)^2}+\dots.$$

6. Prove that

$$\frac{1}{\theta}+\sum_{p=1}^{p=\infty}(-1)^p\left\{\frac{1}{\theta-p\pi}+\frac{1}{\theta+p\pi}\right\}=\operatorname{cosec}\theta.$$

7. Prove that

$$\cot\theta = \frac{1}{\theta}-\frac{1}{2}\tan\frac{\theta}{2}-\frac{1}{2^2}\tan\frac{\theta}{2^2}-\frac{1}{2^3}\tan\frac{\theta}{2^3}-\dots.$$

8. Sum $$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\dots.$$

9. Prove that:

(i) $$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8},$$

(ii) $$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots=\frac{\pi^4}{96}.$$

10. Prove that $$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots=\frac{\pi^2}{12}.$$

11. Prove that

$$1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{10^2}+\dots=\frac{4\pi^2}{27}.$$

12. Sum the infinite series

$$\frac{1}{1\,.\,2}+\frac{1}{2\,.\,4}+\frac{1}{3\,.\,6}+\dots.$$

13. Sum the infinite series

$$\frac{2}{1^4}+\frac{5}{2^4}+\frac{10}{3^4}+\frac{17}{4^4}+\dots.$$

14. Prove that $\Sigma\,\dfrac{1}{r^2s^2}=\dfrac{\pi^4}{120}$, where the summation includes all positive integral values of r and s except that r and s must not be equal.

15. Prove that the sum of the products two together of the reciprocals of the squares of all different positive odd numbers is $\dfrac{\pi^4}{384}$.

16. Find the sum of the series

$$\frac{1}{1^2\,.\,2^2}+\frac{1}{2^2\,.\,3^2}+\frac{1}{3^2\,.\,4^2}+\dots.$$

17. If 2, 3, 5, ... are all the prime numbers, prove that:

$$\text{(i)}\quad \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\ldots=\frac{6}{\pi^2},$$

$$\text{(ii)}\quad \left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\ldots=\frac{15}{\pi^2}.$$

18. Prove that the coefficient of x^n in $(1+x)\left(1+\dfrac{x}{2^2}\right)\left(1+\dfrac{x}{3^2}\right)\ldots$ is $\dfrac{\pi^{2n}}{(2n+1)!}$.

19. When n is large, show that an approximate value of

$$\sum_{p=1}^{p=n}\log\left(1-\frac{1}{2p}\right) \text{ is } \tfrac{1}{2}\log\left(\frac{1}{n\pi}\right).$$

CHAPTER XXIV

THE CUBIC EQUATION. SYMMETRIC FUNCTIONS OF ROOTS OF EQUATIONS

§ 1. Equations involving trigonometrical functions have been solved in many parts of this book, and we have seen how trigonometry may be used to solve the algebraical cubic equation with three real roots (Chap. x, p. 161).

In this chapter we shall first deal with the other cases of the cubic equation; then we shall consider various methods of finding relations between symmetric functions of roots of equations, the methods generally depending on the expansions found in Chapter xx.

§ 2. Solution of the **Cubic Equation.**

Any cubic equation can be reduced to the form

$$x^3 + px = q.$$

I. In Chap. x, p. 161, we solved such an equation by aid of the formula

$$4\cos^3\theta - 3\cos\theta = \cos 3\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If we put $x = k\cos\theta$, we get

$$k^3\cos^3\theta + pk\cos\theta = q,$$

or

$$4\cos^3\theta + \frac{4p}{k^2}\cos\theta = \frac{4q}{k^3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

For (1) and (2) to be the same equation

$$\frac{4p}{k^2} = -3 \quad \text{and} \quad \frac{4q}{k^3} = \cos 3\theta.$$

The method will only apply if p is negative and $\dfrac{4q}{k^3}$ lies between 1 and -1 (including both limits).

II. The corresponding equation

$$4 \cosh^3 u - 3 \cosh u = \cosh 3u$$

enables us to solve the cubic equation if p is negative and $\frac{4q}{k^3}$ is positive and $\nless 1$.

III. Again, the formula

$$4 \sinh^3 u + 3 \sinh u = \sinh 3u$$

enables us to solve the cubic equation for the cases in which p is positive without any restriction as to the value of q.

IV. Thus we can deal with every case except that in which p is negative and $\frac{4q}{k^3}$ is negative and numerically greater than 1 (*i.e.* < -1).

This case can be dealt with by putting $x = -y$, which gives us the equation

$$y^3 + py = -q,$$

and so now comes under II.

In cases II, III, IV we shall get an equation

$$\cosh 3u = \dots \quad \text{or} \quad \sinh 3u = \dots,$$

from which, by the aid of tables, we can find $3u$, and so u. Now if $3a$ is the value of $3u$ found,

the general values of $3u$ are $3a$, $3a + 2\pi i$, $3a + 4\pi i$, ...,

$\therefore$ " " u " a, $a + \frac{2}{3}\pi i$, $a + \frac{4}{3}\pi i$, ...;

therefore the three values of $\cosh u$ are

$$\cosh a, \quad \cosh\left(a + \frac{2\pi i}{3}\right), \quad \cosh\left(a + \frac{4\pi i}{3}\right),$$

and similarly for $\sinh u$.

Therefore in cases II, III, IV one root of the equation is real and two are imaginary.

EXERCISE XXIV. a.

Solve the cubic equations:

1. $x^3 - 3x - 4 = 0$.
3. $x^3 + 3x - 2 = 0$.
2. $x^3 - x - 2 = 0$.
4. $2x^3 - 2x + 3 = 0$.

§ 3. Symmetric Functions of Roots of Equations.

Before reading the rest of the chapter, the student should revise the elementary work on the Theory of Equations to be found in most books on Higher Algebra. The chief knowledge required here is:

(i) If $\alpha, \beta, \gamma, \ldots$ are the roots of the equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots = 0,$$

$$\Sigma \alpha = -\frac{a_1}{a_0}, \ \Sigma \alpha\beta = \frac{a_2}{a_0}, \ \Sigma \alpha\beta\gamma = -\frac{a_3}{a_0}, \text{ etc.}$$

(ii) Familiarity with the transformations of roots of equations.

The reader is advised to work through Miscellaneous Exercises II, p. 170, Nos. 69–75.

§ 4. Relations between the roots of fairly simple equations may often be obtained by elementary methods. The formulae of Chap. VII, § 7, p. 113, are particularly helpful; the following example illustrates their use.

Example i. *Prove that the equation*

$$a^2 \cos^2 \phi + b^2 \sin^2 \phi + 2ga \cos \phi + 2fb \sin \phi + c^2 = 0$$

has four roots, not differing by a multiple of 2π, and that the sum of these roots is a multiple of 2π.

Let $t \equiv \tan \frac{\phi}{2}$, then $\quad \sin \phi = \frac{2t}{1+t^2}, \quad \cos \phi = \frac{1-t^2}{1+t^2}.$

Substitute these values in the equation; it reduces to

$$t^4 (a^2 - 2ga + c^2) + 4fbt^3 + t^2 (4b^2 - 2a^2 + 2c^2) + 4fbt + (a^2 + 2ga + c^2) = 0, \ \ldots\text{(i)}$$

an equation having four roots, say $\phi_1, \phi_2, \phi_3, \phi_4$.

Also $$\tan \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4}{2} = \frac{\Sigma T_1 - \Sigma T_3}{1 - \Sigma T_2 + \Sigma T_4} \qquad (\text{§ 15, p. 244}).$$

From (i) $$\Sigma T_1 - \Sigma T_3 = \frac{-4fb + 4fb}{a^2 - 2ga + c^2} = 0.$$

$$\frac{\phi_1 + \phi_2 + \phi_3 + \phi_4}{2} = n\pi,$$

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 = 2n\pi.$$

EXERCISE XXIV. b.

1. Prove that the equation $\cos 2\theta + a\cos\theta + b\sin\theta + c = 0$ has, in general, four solutions α, β, γ, δ, between 0 and 2π. Prove that $\alpha+\beta+\gamma+\delta$ is a multiple of 2π.

2. Assuming that the values of θ found from the equation $\cos(\theta - a) = c\tan\theta$ are all real, determine the number of roots that lie between 0 and 2π, and prove that the sum of these roots is $2a$ or differs from $2a$ by a multiple of 2π.

3. Prove that, in general, the equation $A\sin^3 x + B\cos^3 x + C = 0$ has six distinct roots, $\alpha_1, \alpha_2, \dots, \alpha_6$, no two of which differ by a multiple of 2π, and that

$$\tan\tfrac{1}{2}(\alpha_1 + \alpha_2 + \dots + \alpha_6) = -\frac{A}{B}.$$

4. If α, β, γ, δ be four roots of the equation $\tan\left(\theta + \frac{\pi}{4}\right) = 3\tan 3\theta$, and no two of them have equal tangents, prove that $\Sigma\tan 2\alpha = \frac{4}{3}$.

5. Prove that the equation $a\sin 2\theta + b\sin\theta + c = 0$ has, in general, four solutions α, β, γ, δ, not differing by multiples of 2π, such that $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.

Also prove that:

(i) $\sin\alpha + \sin\beta + \sin\gamma + \sin\delta = 0$,

(ii) $4\sin\alpha\sin\beta\sin\gamma\sin\delta\,(\Sigma\sin\alpha\sin\beta + 1) = (\Sigma\sin\alpha\sin\beta\sin\gamma)^2$,

(iii) $(\cos\alpha + \cos\beta)(1 + \cos\gamma\cos\delta) + (\cos\gamma + \cos\delta)(1 + \cos\alpha\cos\beta) = 0$.

§ 5. Functions of ratios of angles of the type $\left(\theta + \frac{p\pi}{n}\right)$, including $\frac{p\pi}{n}$, can be deduced from the expansions for $\cos n\theta$, $\sin n\theta$ in Chap. XX, §§ 1–6, or the formula for $\tan n\theta$ in Chap. XIV, § 17.

For instance, the expansion of $\cos n\theta$ (formula (1), p. 320) gives an equation with roots of the type $\cos\left(\theta + \frac{2p\pi}{n}\right)$. See Example iii, below.

The expansion of $\sin n\theta$ (formula (2), p. 321) gives an equation with roots of the same type.

The formula for $\tan n\theta$ gives an equation with roots of the type

$$\tan\left(\theta + \frac{p\pi}{n}\right).$$

Again, formulae (7)–(10) of p. 325 give equations with roots of the type $\sin\left(\theta + \frac{2p\pi}{n}\right)$.

From these equations, by putting $\theta=0$, equations involving functions of $\frac{p\pi}{n}$ are derived.† In these cases it is often necessary to use the fact that $\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin n\theta}{\sin\theta}=\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin n\theta}{n\theta}\times\frac{\theta}{\sin\theta}\times n=n$ (see p. 139).

Example ii. *Prove that:*

(i) $\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)=\frac{3}{4}$,

(ii) $\cos\theta\cos\left(\theta+\frac{2\pi}{3}\right)\cos\left(\theta+\frac{4\pi}{3}\right)=\frac{1}{4}\cos 3\theta.$

$$\cos 3\theta=-3\cos\theta+4\cos^3\theta.$$

$$\therefore\ 4\cos^3\theta-3\cos\theta-\cos 3\theta=0$$

is an equation in $\cos\theta$ whose three roots are $\cos\theta$, $\cos\left(\theta+\frac{2\pi}{3}\right)$, $\cos\left(\theta+\frac{4\pi}{3}\right)$. Hence the results follow.

Example iii. *Prove that, if n is odd:*

(i) $\cos a\cos\left(a+\frac{2\pi}{n}\right)\ldots\cos\left(a+\frac{\overline{n-1}\,2\pi}{n}\right)=\frac{1}{2^{n-1}}\cos na$,

(ii) $\sec a+\sec\left(a+\frac{2\pi}{n}\right)+\ldots+\sec\left(a+\frac{\overline{n-1}\,2\pi}{n}\right)=(-1)^{\frac{n-1}{2}}n\sec na.$

From Chap. xx, § 2, p. 320, if n is odd,

$$2\cos n\theta=(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\frac{n(n-3)}{1.2}(2\cos\theta)^{n-4}-\ldots+(-1)^{\frac{n-1}{2}}n.2\cos\theta.$$

If θ has any one of the n values a, $a+\frac{2\pi}{n}$, $a+\frac{4\pi}{n}$, ..., $a+\frac{2(n-1)\pi}{n}$, then $\cos n\theta$ has the value $\cos na$.

Hence, putting $x=2\cos\theta$, the equation

$$x^n-nx^{n-2}+\frac{n(n-3)}{1.2}x^{n-4}-\ldots+(-1)^{\frac{n-1}{2}}nx-2\cos na=0$$

has roots $\quad 2\cos a,\ \ 2\cos\left(a+\frac{2\pi}{n}\right),\ \ \ldots,\ \ 2\cos\left(a+\frac{\overline{n-1}\,2\pi}{n}\right).$

$$\therefore\ \cos a\cos\left(a+\frac{2\pi}{n}\right)\ldots\cos\left(a+\frac{\overline{n-1}\,2\pi}{n}\right)=\frac{2\cos na}{2^n}=\frac{1}{2^{n-1}}\cos na.$$

And $\frac{1}{2}\left[\sec a+\sec\left(a+\frac{2\pi}{n}\right)+\ldots+\sec\left(a+\frac{\overline{n-1}\,2\pi}{n}\right)\right]=\frac{(-1)^{\frac{n-1}{2}}n}{2\cos na}.$

$$\therefore\ \sec a+\sec\left(a+\frac{2\pi}{n}\right)+\ldots+\sec\left(a+\frac{\overline{n-1}\,2\pi}{n}\right)=(-1)^{\frac{n-1}{2}}n\sec na.$$

† Compare Chap. XXII.

Example iv. *Prove that, if n is odd:*

(i) $$\tan\alpha\tan\left(\alpha+\frac{\pi}{n}\right)\tan\left(\alpha+\frac{2\pi}{n}\right)\ldots\tan\left(\alpha+\frac{\overline{n-1}\,\pi}{n}\right)=(-1)^{\frac{n-1}{2}}\tan n\alpha.$$

(ii) $$\tan\frac{\pi}{n}\tan\frac{2\pi}{n}\ldots\tan\frac{\overline{n-1}\,\pi}{n}=(-1)^{\frac{n-1}{2}}n.$$

From p. 244, if n is odd,

$$\tan n\theta=\frac{n\tan\theta-{}_nC_3\tan^3\theta+\ldots+(-1)^{\frac{n-1}{2}}\tan^n\theta}{1-{}_nC_2\tan^2\theta+\ldots+(-1)^{\frac{n-1}{2}}n\tan^{n-1}\theta}.$$

If θ has any one of the n values α, $\alpha+\frac{\pi}{n}$, $\alpha+\frac{2\pi}{n}$, …, $\alpha+\frac{(n-1)\pi}{n}$, then $\tan n\theta$ has the value $\tan n\alpha$.

Hence, putting $x=\tan\theta$,

$$(-1)^{\frac{n-1}{2}}x^n-\ldots-{}_nC_3x^3+nx-\tan n\alpha\,\{(-1)^{\frac{n-1}{2}}nx^{n-1}-\ldots-{}_nC_2x^2+1\}=0$$

is an equation whose n roots are $\tan\alpha$, $\tan\left(\alpha+\frac{\pi}{n}\right)$, …, $\tan\left(\alpha+\frac{\overline{n-1}\,\pi}{n}\right)$

$$\therefore\ \tan\alpha\tan\left(\alpha+\frac{\pi}{n}\right)\tan\left(\alpha+\frac{2\pi}{n}\right)\ldots\tan\left(\alpha+\frac{\overline{n-1}\,\pi}{n}\right)=(-1)^{\frac{n-1}{2}}\tan n\alpha.$$

If we put $\alpha=0$, we get

$$\tan\frac{\pi}{n}\tan\frac{2\pi}{n}\ldots\tan\frac{\overline{n-1}\,\pi}{n}=(-1)^{\frac{n-1}{2}}\operatorname*{Lt}_{\alpha\to0}\frac{\tan n\alpha}{\tan\alpha}=(-1)^{\frac{n-1}{2}}n\quad\text{(see p. 139).}$$

This last result could have been obtained as follows:

$$\tan n\theta=0\quad\text{when}\quad\theta=0,\ \frac{\pi}{n},\ \frac{2\pi}{n},\ \ldots,\ \frac{\overline{n-1}\,\pi}{n}.$$

Hence, putting $\tan\theta=x$ in the formula for $\tan n\theta$, when n is odd,

$$nx-{}_nC_3x^3+\ldots+(-1)^{\frac{n-1}{2}}x^n=0$$

is an equation whose roots are

$$\tan 0,\ \tan\frac{\pi}{n},\ \tan\frac{2\pi}{n},\ \ldots,\ \tan\frac{\overline{n-1}\,\pi}{n}.$$

$$\therefore\ n-{}_nC_3x^2+\ldots+(-1)^{\frac{n-1}{2}}x^{n-1}=0$$

has roots
$$\tan\frac{\pi}{n},\ \tan\frac{2\pi}{n},\ \ldots,\ \tan\frac{\overline{n-1}\,\pi}{n}.$$

$$\therefore\ \tan\frac{\pi}{n}\tan\frac{2\pi}{n}\ldots\tan\frac{\overline{n-1}\,\pi}{n}=(-1)^{\frac{n-1}{2}}n.$$

EXERCISE XXIV. c.

1. Prove that:

(i) $\sin\theta+\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta+\frac{4\pi}{3}\right)=0,$

(ii) $4\sin\theta\sin\left(\theta+\frac{2\pi}{3}\right)\sin\left(\theta+\frac{4\pi}{3}\right)=-\sin 3\theta.$

2. Prove that:

(i) $\tan\theta+\tan\left(\theta+\frac{\pi}{3}\right)+\tan\left(\theta+\frac{2\pi}{3}\right)=3\tan 3\theta,$

(ii) $\tan\theta\tan\left(\theta+\frac{\pi}{3}\right)\tan\left(\theta+\frac{2\pi}{3}\right)=-\tan 3\theta,$

(iii) $\tan^2\theta+\tan^2\left(\theta+\frac{\pi}{3}\right)+\tan^2\left(\theta+\frac{2\pi}{3}\right)=6+9\tan^2 3\theta.$

3. Prove that, if n be odd:

(i) $\tan\theta+\tan\left(\theta+\frac{\pi}{n}\right)+\tan\left(\theta+\frac{2\pi}{n}\right)+\ldots+\tan\left(\theta+\frac{\overline{n-1}\,\pi}{n}\right)=n\tan n\theta,$

(ii) $\cot\theta+\cot\left(\theta+\frac{\pi}{n}\right)+\cot\left(\theta+\frac{2\pi}{n}\right)+\ldots+\cot\left(\theta+\frac{\overline{n-1}\,\pi}{n}\right)=n\cot n\theta.$

4. Prove that:

(i) $\cot^2\theta+\cot^2\left(\theta+\frac{\pi}{n}\right)+\cot^2\left(\theta+\frac{2\pi}{n}\right)+\ldots+\cot^2\left(\theta+\frac{\overline{n-1}\,\pi}{n}\right)$
$=n^2\cot^2 n\theta+n(n-1),$

(ii) $\tan^2\theta+\tan^2\left(\theta+\frac{\pi}{n}\right)+\tan^2\left(\theta+\frac{2\pi}{n}\right)+\ldots+\tan^2\left(\theta+\frac{\overline{n-1}\,\pi}{n}\right)$
$=n^2\cot^2\left(\frac{n\pi}{2}+n\theta\right)+n(n-1).$

5. If n be even, prove that:

(i) $\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)\cos\left(\theta+\frac{4\pi}{n}\right)\ldots\cos\left(\theta+\frac{\overline{n-1}\,2\pi}{n}\right)$
$=\frac{1}{2^{n-1}}\left[(-1)^{\frac{n}{2}}-\cos n\theta\right],$

(ii) $\sec\theta+\sec\left(\theta+\frac{2\pi}{n}\right)+\sec\left(\theta+\frac{4\pi}{n}\right)+\ldots+\sec\left(\theta+\frac{\overline{n-1}\,2\pi}{n}\right)=0.$

6. If n be odd, prove that:

(i) $\sec\theta\sec\left(\theta+\frac{2\pi}{n}\right)\ \ldots\ \sec\left(\theta+\frac{\overline{n-1}\,2\pi}{n}\right)=2^{n-1}\sec n\theta,$

(ii) $\sec^2\theta+\sec^2\left(\theta+\frac{2\pi}{n}\right)+\ldots+\sec^2\left(\theta+\frac{\overline{n-1}\,2\pi}{n}\right)=n^2\sec^2 n\theta,$

(iii) $\tan^2\theta+\tan^2\left(\theta+\frac{2\pi}{n}\right)+\ldots+\tan^2\left(\theta+\frac{\overline{n-1}\,2\pi}{n}\right)=n(n\sec^2 n\theta-1).$

7. Prove that:

(i) $$2^{n-1}\sin\theta\sin\left(\theta+\frac{2\pi}{n}\right)\ \dots\ \sin\left(\theta+\frac{\overline{n-1}\,2\pi}{n}\right)$$

$$=(-1)^{\frac{n-1}{2}}\sin n\theta, \text{ when } n \text{ is odd,}$$

$$\text{and }=(-1)^{\frac{n}{2}}(1-\cos n\theta), \text{ when } n \text{ is even.}$$

(ii) $$\operatorname{cosec}^2\theta+\operatorname{cosec}^2\left(\theta+\frac{2\pi}{n}\right)+\dots+\operatorname{cosec}^2\left(\theta+\frac{\overline{n-1}\,2\pi}{n}\right)$$

$$=n^2\operatorname{cosec}^2 n\theta, \text{ when } n \text{ is odd,}$$

$$\text{and}=\tfrac{1}{2}n^2\operatorname{cosec}^2\frac{n\theta}{2}, \text{ when } n \text{ is even.}$$

8. Prove that, when n is odd, the sum of the products taken two together of the $(n-1)$ quantities $\tan\frac{\pi}{n}$, $\tan\frac{2\pi}{n}$, ..., $\tan\frac{\overline{n-1}\,\pi}{n}$ is $\frac{n(1-n)}{2}$.

9. Prove that, when n is even:

(i) $$\tan\frac{\pi}{2n}+\tan\frac{3\pi}{2n}+\dots+\tan\frac{\overline{2n-1}\,\pi}{2n}=0,$$

(ii) $$\tan\frac{\pi}{2n}\tan\frac{3\pi}{2n}\ \dots\ \tan\frac{\overline{2n-1}\,\pi}{2n}=(-1)^{\frac{n}{2}}.$$

10. Prove that:

(i) $$\sum_{p=1}^{p=n-1}\sec^2\frac{p\pi}{n}=n^2-1, \text{ when } n \text{ is odd.}$$

(ii) $$\sum_{p=1}^{p=n-1}\operatorname{cosec}^2\frac{p\pi}{n}=\frac{n^2-1}{3}.$$

(iii) $$\sum_{p=1}^{p=n-1}\operatorname{cosec}^2\frac{p\pi}{2n}=\frac{2(n^2-1)}{3}.$$

11. Prove that:

(i) $$\sin^2\frac{\pi}{11}+\sin^2\frac{2\pi}{11}+\sin^2\frac{3\pi}{11}+\sin^2\frac{4\pi}{11}+\sin^2\frac{5\pi}{11}=\frac{11}{4},$$

(ii) $$\cot^2\frac{\pi}{11}+\cot^2\frac{2\pi}{11}+\cot^2\frac{3\pi}{11}+\cot^2\frac{4\pi}{11}+\cot^2\frac{5\pi}{11}=15.$$

12. Prove that the values of x which satisfy the equation

$$1-nx-\frac{n(n-1)}{\lfloor 2}x^2+\frac{n(n-1)(n-2)}{\lfloor 3}x^3+\dots+(-1)^{\frac{1}{2}n(n+1)}x^n=0$$

are given by $x=\tan\frac{(4p+1)\pi}{4n}$, where p is any integer

13. Prove that $$\tan\frac{\pi}{4n}\tan\frac{3\pi}{4n}\ \dots\dots\ \tan\frac{(2n-1)\pi}{4n}=1.$$

§ 6. The evaluation of symmetric functions of the trigonometrical ratios of angles such as $\frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \ldots$ for special values of n may be effected in various ways, most of which ultimately depend on forming an equation of which the ratios are the roots. An example will illustrate some of these ways.

Example v. *Prove that:*

$$\text{(i)}\quad \cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{6\pi}{9}+\cos\frac{8\pi}{9}=-\frac{1}{2},$$

$$\text{(ii)}\quad \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{6\pi}{9}\cos\frac{8\pi}{9}=\frac{1}{16}.$$

First solution.

[This is not very suitable except in the case of small denominators; even with as large a number as 9 it is clumsy.]

If θ be any of the angles $0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{6\pi}{9}, \frac{8\pi}{9}$, then $9\theta=2n\pi$.

$$\therefore\ \cos 4\theta=\cos 5\theta=\cos 2\theta\cos 3\theta-\sin 2\theta\sin 3\theta,$$

$$\therefore\ 2\cos^2 2\theta-1=(2\cos^2\theta-1)(-3\cos\theta+4\cos^3\theta)-2\sin\theta\cos\theta\,(3\sin\theta-4\sin^3\theta).$$

Let $x\equiv\cos\theta$, then $\sin^2\theta=1-x^2$.

$$\therefore\ 2(2x^2-1)^2-1=(2x^2-1)(-3x+4x^3)-2x(1-x^2)(4x^2-1),$$

which gives the equation

$$16x^5-8x^4-20x^3+8x^2+5x-1=0.$$

$$\therefore\ \cos 0+\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{6\pi}{9}+\cos\frac{8\pi}{9}=\frac{1}{2},$$

$$\therefore\ \cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{6\pi}{9}+\cos\frac{8\pi}{9}=-\frac{1}{2},$$

and
$$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{6\pi}{9}\cos\frac{8\pi}{9}=\frac{1}{16},$$

since $\cos 0=1$.

Second solution.

By quoting the series for $\cos n\theta$ in Chap. xx, § 2, p. 320,

$$2\cos 9\theta=(2\cos\theta)^9-9(2\cos\theta)^7+\ldots+9(2\cos\theta).$$

If $\cos 9\theta=1$, $9\theta=2n\pi$,

$$\therefore\ \theta=0, \frac{2\pi}{9}, \frac{4\pi}{9}, \ldots, \frac{16\pi}{9}.$$

Hence, if $x\equiv 2\cos\theta$, the equation

$$x^9-9x^7+\ldots+9x=2$$

has for its roots the values of $2\cos\theta$ when θ has the 9 values

$$0, \frac{2\pi}{9}, \frac{4\pi}{9}, \ldots, \frac{16\pi}{9}.$$

Hence
$$\sum_{p=0}^{p=8} \cos\frac{2p\pi}{9}=0.$$
$$\therefore\ \cos 0+2\sum_{p=1}^{p=4}\cos\frac{2p\pi}{9}=0,$$
$$\therefore\ \cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{6\pi}{9}+\cos\frac{8\pi}{9}=-\frac{1}{2}.$$
Again
$$2^9\prod_{p=0}^{p=8}\cos\frac{2p\pi}{9}=2.$$
$$\therefore\ \cos 0\prod_{p=1}^{p=4}\cos^2\frac{2p\pi}{9}=\frac{1}{2^8},$$
$$\therefore\ \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{6\pi}{9}\cos\frac{8\pi}{9}=\sqrt{\frac{1}{2^8}}=\frac{1}{16},$$
since two of the factors are negative and two positive.

Third solution.

The required equation is here found by the use of complex numbers.

Let $y=(1,\ \theta)$,
$$\therefore\ y^9=(1, 9\theta).$$
Hence $y^9=1$, if θ is $0, \frac{2\pi}{9}, \frac{4\pi}{9}, \dots, \frac{16\pi}{9}$.

Therefore the equation
$$y^9-1=0,$$
$$i.e.\ (y-1)(y^8+y^7+\dots+1)=0,$$
has for its roots the 9 values of $(1,\ \theta)$ corresponding to the 9 values
$$0, \frac{2\pi}{9}, \frac{4\pi}{9}, \dots, \frac{16\pi}{9}$$
of θ. Also the root $y=1$ obviously corresponds to the value $\theta=0$.

Now let $y+\frac{1}{y}=2x$,
$$\therefore\ x=\cos\theta,$$
and $\quad y^2+\frac{1}{y^2}=4x^2-2, \quad y^3+\frac{1}{y^3}=8x^3-6x, \quad y^4+\frac{1}{y^4}=16x^4-16x^2+2.$

Hence the equation
$$y^8+y^7+\dots+1=0,$$
$$i.e.\ \left(y^4+\frac{1}{y^4}\right)+\left(y^3+\frac{1}{y^3}\right)+\left(y^2+\frac{1}{y^2}\right)+\left(y+\frac{1}{y}\right)+1=0,$$
becomes
$$16x^4+8x^3-12x^2-4x+1=0,$$
which is an equation in x satisfied by $\cos\theta$, where θ has the values
$$\frac{2\pi}{9}\left(\text{or } \frac{16\pi}{9}\right), \frac{4\pi}{9}\left(\text{or } \frac{14\pi}{9}\right), \frac{6\pi}{9}\left(\text{or } \frac{12\pi}{9}\right), \frac{8\pi}{9}\left(\text{or } \frac{10\pi}{9}\right).$$
Hence
$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{6\pi}{9}+\cos\frac{8\pi}{9}=-\frac{1}{2},$$
and
$$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{6\pi}{9}\cos\frac{8\pi}{9}=\frac{1}{16}.$$

EXERCISE XXIV. d.

1. (i) Prove that $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$ are the roots of the equation

$$8x^3-4x^2-4x+1=0.$$

(ii) Prove that $\sec^2\frac{\pi}{7}$, $\sec^2\frac{3\pi}{7}$, $\sec^2\frac{5\pi}{7}$ are the roots of the equation

$$x^3-24x^2+80x-64=0.$$

(iii) Prove that $\tan^2\frac{\pi}{7}$, $\tan^2\frac{3\pi}{7}$, $\tan^2\frac{5\pi}{7}$ are the roots of the equation

$$x^3-21x^2+35x-7=0.$$

(iv) Prove that

$$\left(1+2\cos\frac{\pi}{7}\right)\left(1+2\cos\frac{3\pi}{7}\right)\left(1+2\cos\frac{5\pi}{7}\right)+1=0.$$

(v) Prove that $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, $\cos\frac{6\pi}{7}$ are the roots of the equation

$$8x^3+4x^2-4x-1=0.$$

2. Show that

$$\sin\frac{3\pi}{7}+\sin\frac{2\pi}{7}-\sin\frac{\pi}{7}=4\sin\frac{3\pi}{7}\sin\frac{2\pi}{7}\sin\frac{\pi}{7}=\frac{1}{2}\sqrt{7}.$$

3. (i) Find a rational integral equation in $\tan\theta$ which is satisfied by $\tan\frac{\pi}{7}$. What are the other roots?

(ii) Prove that $\tan^2\frac{\pi}{7}$, $\tan^2\frac{2\pi}{7}$, $\tan^2\frac{4\pi}{7}$ satisfy the equation

$$z^3-21z^2+35z-7=0.$$

(iii) Hence, or otherwise, show that the roots of the equation

$$x^3+x^2\sqrt{7}-7x+\sqrt{7}=0$$

are $\tan\frac{\pi}{7}$, $\tan\frac{2\pi}{7}$, $\tan\frac{4\pi}{7}$.

4. Expand $\frac{\cos 7\theta}{\cos\theta}$ in powers of $\sin^2\theta$.

Prove that $$\sin^2\frac{\pi}{14}+\sin^2\frac{3\pi}{14}+\sin^2\frac{5\pi}{14}=\frac{5}{4},$$

and that $$\cot^2\frac{\pi}{14}+\cot^2\frac{3\pi}{14}+\cot^2\frac{5\pi}{14}=21.$$

5. Find an equation whose roots are the tangents of θ, 2θ, 4θ, 5θ, 7θ and 8θ, where $\theta=20°$; hence show that

(i) $\tan 20°\tan 40°\tan 80°=\sqrt{3}$,

(ii) $\tan^2 20°+\tan^2 40°+\tan^2 80°=33$.

6. Prove that $$\cos^2\frac{\pi}{9}+\cos^2\frac{2\pi}{9}+\cos^2\frac{3\pi}{9}+\cos^2\frac{4\pi}{9}=\frac{7}{4}.$$

7. Prove that $x=2\cos\frac{\pi}{9}$ is a root of the equation
$$x^6-6x^4+9x^2-1=0,$$
and write down the other roots.

Hence find the cubic equation whose roots are $\cos^2\frac{\pi}{9}$, $\cos^2\frac{4\pi}{9}$, $\cos^2\frac{7\pi}{9}$.

8. (i) If $\alpha=\frac{\pi}{9}$, prove that
$$\sec^2\alpha+\sec^2 2\alpha+\sec^2 4\alpha=36=3\,(\operatorname{cosec}^2\alpha+\operatorname{cosec}^2 2\alpha+\operatorname{cosec}^2 4\alpha).$$

(ii) Prove that $$\tan^2\frac{\pi}{9}+\tan^2\frac{4\pi}{9}+\tan^2\frac{7\pi}{9}=33.$$

9. Prove that tan 10°, tan 70°, tan 130° are the three roots of the equation
$$\sqrt{3}\,x^3-3x^2-3\sqrt{3}\,x+1=0.$$

10. Prove that $2\cos\frac{2\pi}{11}$ is a root of the equation
$$y^5+y^4-4y^3-3y^2+3y+1=0,$$
and show that

(i) $\cos\frac{2\pi}{11}+\cos\frac{4\pi}{11}+\cos\frac{6\pi}{11}+\cos\frac{8\pi}{11}+\cos\frac{10\pi}{11}=-\frac{1}{2}$,

(ii) $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}=\frac{1}{2}$.

11. Prove that $$\sum_{p=1}^{p=10}\operatorname{cosec}^2\frac{p\pi}{11}=40.$$

12. Prove that:

(i) $\sin^2\frac{\pi}{11}+\sin^2\frac{2\pi}{11}+\sin^2\frac{3\pi}{11}+\sin^2\frac{4\pi}{11}+\sin^2\frac{5\pi}{11}=\frac{11}{4}$,

(ii) $32\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}=1.$

13. Prove that $\tan^2\frac{\pi}{11}$, $\tan^2\frac{2\pi}{11}$, $\tan^2\frac{3\pi}{11}$, $\tan^2\frac{4\pi}{11}$, $\tan^2\frac{5\pi}{11}$ are the roots of the equation
$$x^5-55x^4+330x^3-462x^2+165x-11=0.$$

Hence find the values of $\sum_{p=1}^{p=5}\cot^2\frac{p\pi}{11}$ and $\sum_{p=1}^{p=5}\sec^2\frac{p\pi}{11}$.

14. If $\theta=\frac{2\pi}{11}$, prove that
$$\cos^5\theta+\cos^5 2\theta+\cos^5 3\theta+\cos^5 4\theta+\cos^5 5\theta=-\tfrac{1}{2}.$$

15. Prove that $$\sum_{p=1}^{p=12} \sec^2\frac{p\pi}{13} = 168.$$

16. Prove that $$\prod_{p=1}^{p=9} \cos\frac{p\pi}{19} = \frac{1}{512}.$$

17. Prove that $$\sum_{p=1}^{p=8} \sec^2\frac{2p\pi}{17} = 144.$$

18. Prove that

$$\operatorname{cosec}^2\frac{\pi}{20} + \operatorname{cosec}^2\frac{3\pi}{20} + \operatorname{cosec}^2\frac{7\pi}{20} + \operatorname{cosec}^2\frac{9\pi}{20} = 48.$$

MISCELLANEOUS EXERCISES IV

1. Show that, if the modulus of z be unity, the points representing $\sqrt{\dfrac{1+z}{1-z}}$ lie on one or other of two perpendicular lines.

2. The complex numbers z, z_1, z_2 are represented in an Argand diagram by a variable point P and fixed points A and B respectively. Prove that the variable point P moves on a straight line if $\log \dfrac{z-z_1}{z-z_2}$ is (i) purely real, or (ii) purely imaginary.

What angles do the lines corresponding to (i) and (ii) make with AB?

3. If $\cos\alpha+\cos\beta+\cos\gamma=0$ and $\sin\alpha+\sin\beta+\sin\gamma=0$, prove that

$$\Sigma \cos n\alpha=0 \text{ and } \Sigma \sin n\alpha=0,$$

provided n is an integer prime to 3.

4. If $\cos\theta=\dfrac{1}{2}\left(u+\dfrac{1}{u}\right)$, prove that

$$\sin\theta=\pm\frac{1}{2}\left(u-\frac{1}{u}\right).$$

What does this tell us about u?

5. If $\cos(\alpha+i\beta)=\exp(i\theta)$, prove that

$$\sin\theta=\pm\sin^2\alpha=\pm\sinh^2\beta.$$

6. If $\cos(\alpha+i\beta)=x+iy=(r,\theta)$, prove that:

(i) $\dfrac{x^2}{\cos^2\alpha}-\dfrac{y^2}{\sin^2\alpha}=1$, (iii) $\tan\theta=-\tan\alpha\tanh\beta$,

(ii) $\dfrac{x^2}{\cosh^2\beta}+\dfrac{y^2}{\sinh^2\beta}=1$, (iv) $r^2=\frac{1}{2}(\cosh 2\beta+\cos 2\alpha)$.

7. If $\tan\dfrac{\alpha}{2}=\tanh\dfrac{u}{2}$, where α is acute (positive or negative), prove that

$$u=\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

8. Obtain the values of $(1+e^{i\theta})^{\frac{1}{2}}$ each in the form $a+ib$, where a and b are real.

9. Express $\cosh^{-1}(\cos x+i\sin x)$ in the form A + iB, when x is acute.

10. Prove that $-\dfrac{i}{2}\log\dfrac{a}{x}$ is a value of $\operatorname{Tan}^{-1}\left(i\dfrac{x-a}{x+a}\right)$.

11. Prove that, if $a+ib=q^{x+iy}$, then $\dfrac{y}{x}=\dfrac{2\tan^{-1}\dfrac{b}{a}}{\log(a^2+b^2)}$.

12. If $\log\sin(\theta+i\phi)=\alpha+i\beta$, prove that $2e^{2\alpha}=\cosh 2\phi-\cos 2\theta$.

13. If $\sin(x+iy)\sin(\alpha+i\beta)=1$, show that

$$\tan\alpha=\pm\frac{\sin x}{\sinh y}\quad\text{and}\quad\tanh\beta=\mp\frac{\cos x}{\cosh y}.$$

14. Taking $\cos x$ to be defined by the exponential expression for it, prove that $\cos x=0$ for a real positive value of x which is less than 1·6.

15. If $\phi+i\psi=\cosh^{-1}\dfrac{x+iy}{c}$, where ϕ and ψ are real and x, y are the coordinates of a real point, find the curves $\phi=$a constant, $\psi=$a constant.

16. If $\sin(\xi+i\eta)=k\sin x$, when $k>1$, find how ξ and η vary as x varies from 0 to π.

17. Show that the values of i^i can be arranged so that they form a geometrical progression.

18. Prove that the numerical value of $\dfrac{1}{x}\sin x$ is always less than unity when x is real and finite, and that the numerical value of $\dfrac{1}{x}\sinh x$ is always greater than unity under the same conditions.

Hence show that the equation $\tan z=az$, where a is real, cannot be satisfied by any complex quantity whose real and imaginary parts are both finite.

19. Prove that the roots of the equation $(1+z)^n=(1-z)^n$ are the values of $i\tan\dfrac{r\pi}{n}$, where r may have all integral values from 0 to $(n-1)$, omitting $\dfrac{n}{2}$ if n is even.

20. Determine the roots of the equation $z^n=(z+1)^n$ and show that, if the points representing the roots are marked in an Argand diagram, then these points are collinear.

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21. Prove that, if h and k are small:

(i) $\dfrac{\sin(x+h)-\sin x}{\sin(x+k)-\sin x}=\dfrac{h}{k}$ approximately,

(ii) $\dfrac{\tan(x+h)-\tan x}{\tan(x+k)-\tan x}=\dfrac{h}{k}$ approximately,

(iii) $\dfrac{\log\cos(x+h)-\log\cos x}{\log\cos(x+k)-\log\cos x}=\dfrac{h}{k}$ approximately.

What significance have these results in the compiling of tables?

22. Evaluate $$\operatorname*{Lt}_{y \to x} \frac{x \sin y - y \sin x}{x \cos y - y \cos x}.$$

23. Verify the following construction for finding approximately the length of an arc AB of a circle when the chord AB is less than half the radius. Join BA and produce it to D making AD$=\frac{1}{2}$AB. With centre D and radius DB describe the arc BC cutting at C the tangent AC to the arc at A. Then the length of the arc AB$=$AC.

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24. Prove that $\dfrac{\sin 9\theta}{\sin \theta}=(x^2-1)(x^6-6x^4+9x^2-1)$, where $x=2\cos\theta$.

25. Prove that $\dfrac{1+\cos 9\theta}{1+\cos\theta}=(x^4-x^3-3x^2+2x+1)^2$, where $x=2\cos\theta$.

26. Prove that $\dfrac{1+\cos 11\theta}{1+\cos\theta}$ is the square of a rational function of $\cos\theta$ and find this function.

27. Sum to n terms $$\frac{\cos 2x}{\sin 3x}+\frac{\cos 6x}{\sin 9x}+\frac{\cos 18x}{\sin 27x}+\dots.$$

28. Prove that $$\tan^{-1}\frac{3}{2n^2+4}=\tan^{-1}\frac{n-1}{n+2}-\tan^{-1}\frac{n-2}{n+1}.$$
Hence find the sum to n terms of the series
$$\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{4}+\tan^{-1}\tfrac{3}{22}+\tan^{-1}\tfrac{1}{12}+\dots.$$

29. Sum to n terms
$$\tan^2\theta+4\tan^2 2\theta+4^2\tan^2 2^2\theta+\dots.$$

30. Sum to n terms
$$\cosh\frac{\theta}{2}+2\cosh\frac{\theta}{2}\cosh\frac{\theta}{2^2}+\dots+2^{n-1}\cosh\frac{\theta}{2}\cosh\frac{\theta}{2^2}\dots\cosh\frac{\theta}{2^n}.$$

31. Find the sum of the series
$$\cos a-c_1\cos(a+b)+c_2\cos(a+2b)-c_3\cos(a+3b)+\dots+(-1)^n\cos(a+nb),$$
where $1, c_1, c_2, \dots, 1$ are the coefficients in the expansion of $(1+x)^n$.

32. If $$S=\quad nc\sin\theta+\frac{n(n-1)}{1.2}c^2\sin 2\theta+\dots \text{ to } n \text{ terms}$$
and $$C=1+nc\cos\theta+\frac{n(n-1)}{1.2}c^2\cos 2\theta+\dots \text{ to } (n+1) \text{ terms},$$
prove that $$S^2+C^2=(1+2c\cos\theta+c^2)^n.$$

33. In any triangle, prove that, when n is a positive integer,
$$c^n=a^n\cos nB+na^{n-1}b\cos(\overline{n-1}\,B-A)+\frac{n(n-1)}{1.2}a^{n-2}b^2\cos(\overline{n-2}\,B-2A)+\dots.$$

34. Prove that, if n is an even integer,

$$\cos\frac{2a\pi}{nc}-(n-1)\cos\frac{2(a+c)\pi}{nc}+\frac{(n-1)(n-2)}{1.2}\cos\frac{2(a+2c)\pi}{nc}-\ldots$$
$$+(-1)^{n-1}\cos\frac{2(a+nc-c)\pi}{nc}=(-1)^{\frac{n}{2}}2^{n-1}\left(\sin\frac{\pi}{n}\right)^{n-1}\sin\frac{(2a-c)\pi}{nc}.$$

35. Sum the series:

(i) $n\cosh u+\dfrac{n(n-1)}{1.2}\cosh 2u+\ldots,$

(ii) $\sinh u+n\sinh 2u+\dfrac{n(n-1)}{1.2}\sinh 3u+\ldots.$

36. Find $S_n=\sin x+\sin 2x+\sin 3x+\ldots$ to n terms, and prove that

$$\underset{n\to\infty}{\mathrm{Lt}}\frac{S_1+S_2+\ldots+S_n}{n}=\tfrac{1}{2}\cot\tfrac{1}{2}x.$$

37. Sum the series

$$\frac{n}{2}+(n-1)\cos\theta+(n-2)\cos 2\theta+\ldots+\cos(n-1)\theta.$$

Prove that if $\theta=\dfrac{\pi}{n}$ the limiting value of $\dfrac{1}{n^2}$ times the sum of the series, as n approaches infinity, is $\dfrac{2}{\pi^2}$.

38. Find C_n, the sum to n terms of the series $\cos\theta-\cos 3\theta+\cos 5\theta-\ldots$, and show that, if $-\frac{1}{4}\pi<\theta<\frac{1}{4}\pi$, C_n is positive and less then $\sqrt{2}$.

Prove also that, for such values of θ, the sum of n terms of the series

$$\sin\theta-\tfrac{1}{3}\sin 3\theta+\tfrac{1}{5}\sin 5\theta-\ldots \text{ is } \int_0^\theta\frac{1+(-1)^{n-1}\cos 2nx}{2\cos x}\,dx.$$

39. Prove that

$$\sin^4\frac{\pi}{2n}+\sin^4\frac{2\pi}{2n}+\ldots+\sin^4\frac{n\pi}{2n}=\frac{3n+4}{8}.$$

40. Sum to n terms

$$1^2\cos a-2^2\cos 2a+3^2\cos 3a-\ldots.$$

41. Sum the series

$$1.2\sin\frac{\pi}{n}+2.3\sin\frac{2\pi}{n}+\ldots+(n-1)n\sin\frac{(n-1)\pi}{n}.$$

42. If $P=\sum_{p=0}^{p=n}\operatorname{cosec}2^{p+1}\theta$ and $Q=\sum_{p=0}^{p=n}\tan 2^{p-1}\theta\sec 2^p\theta$,

prove that $$8P-Q=\cot\frac{\theta}{2}+\cot\theta-\cot 2^n\theta-\cot 2^{n+1}\theta.$$

43. Sum to n terms

$$\log\left(x^2 - 2x\cos\frac{\theta}{n}+1\right)+\log\left(x^2-2x\cos\frac{\theta+2\pi}{n}+1\right)$$
$$+\log\left(x^2-2x\cos\frac{\theta+4\pi}{n}+1\right)+,\ldots$$

44. Sum to n terms the series whose rth term is

$$\log\left(\frac{\cos 2r\theta+\sin\theta}{\cos 2r\theta-\sin\theta}\right).$$

45. If n be even, prove that

$$\tan\theta+\tan\left(\theta+\frac{\pi}{n}\right)+\tan\left(\theta+\frac{2\pi}{n}\right)+\ldots+\tan\left(\theta+\frac{\overline{n-1}\pi}{n}\right)=-n\cot n\theta.$$

46. Express $\cos n\theta\sec^n\theta$ in a series of ascending powers of $\tan\theta$, where n is a positive integer.

47. Find the coefficient of x^n in the expansion of $(1+\lambda x+x^2)^n$, where n is a positive integer; and, if $\lambda=2\cos\theta$, deduce that

$$c_0^2+c_1^2\cos 2\theta+c_2^2\cos 4\theta+\ldots+c_n^2\cos 2n\theta$$
$$=\underline{|n}\cos n\theta\left\{\frac{(2\cos\theta)^n}{\underline{|n}}+\frac{(2\cos\theta)^{n-2}}{1^2\,\underline{|n-2}}+\frac{(2\cos\theta)^{n-4}}{1^2.\,2^2\,\underline{|n-4}}+\ldots\right\},$$

where the indices of $\cos\theta$ are positive numbers or zero and

$$(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_nx^n.$$

48. Prove that

$$\frac{\sin n\theta}{\sin\theta}=\frac{\cos n\theta}{\cos\theta}+\frac{\cos(n-1)\theta}{\cos^2\theta}+\ldots+\frac{\cos\theta}{\cos^n\theta}$$
$$=\cos(n-1)\theta+\cos(n-2)\theta\cos\theta+\ldots+\cos\theta\cos^{n-2}\theta+\cos^{n-1}\theta.$$

49. Sum to n terms the series whose rth term is

$$\sin\frac{r(r+1)}{2}\alpha\sin\frac{(r+1)(r+2)}{2}\alpha\sin(r+1)\alpha.$$

50. Sum to n terms the series whose rth term is

$$\frac{\cot(r+1)\theta}{1-\cos^2(r+1)\theta\sec^2\theta}.$$

51. Sum to n terms the series whose rth term is

$$\frac{\sin\alpha\sin r\alpha}{(\cos\alpha+\cos r\alpha)^2}.$$

52. Prove that

$$\frac{2n\tan 2n\theta}{\sin 2\theta}=\sum_{r=1}^{r=n}\frac{1}{\sin^2\frac{(2r-1)\pi}{4n}-\sin^2\theta}.$$

53. Prove that

$$\frac{x^{n-1}}{x^{2n}-2x^n\cos n\theta+1}=\frac{1}{n\sin n\theta}\sum_{r=0}^{r=n-1}\frac{\sin\left(\theta+\frac{2r\pi}{n}\right)}{x^2-2x\cos\left(\theta+\frac{2r\pi}{n}\right)+1}.$$

54. Show that

$$2n\sec n\theta=\Sigma\frac{\sin\frac{r\pi}{2n}}{\sin\frac{1}{2}\left(\frac{r\pi}{2n}-\theta\right)\sin\frac{1}{2}\left(\frac{r\pi}{2n}+\theta\right)},$$

where r has the values 1, 5, 9, ..., $\overline{4n-3}$.

55. Show that, if $0<m<n$ and $(m+n)$ is odd,

$$\frac{\sin mx}{\sin nx}=\frac{1}{n}\sum_{r=0}^{r=n-1}(-1)^r\sin\frac{mr\pi}{n}\operatorname{cosec}\left(x-\frac{r\pi}{n}\right).$$

56. The circumference of a given circle of radius r is divided into n equal parts at A_0, A_1, A_2, ..., A_{n-1}; if the distances from A_0 of the points A_1, A_2, ..., A_{n-1} are denoted by a_1, a_2, ..., a_{n-1}, show that

$$a_1a_2+a_2a_3+\ldots+a_{n-2}a_{n-1}=2nr^2\cos\frac{\pi}{n}.$$

57. Prove that the sum of the squares of the reciprocals of the lines joining one vertex of a regular polygon of n sides to the other $(n-1)$ vertices is $\frac{n^2-1}{12r^2}$, where r is the radius of the circumscribing circle.

58. A regular polygon of n sides is inscribed in a circle of radius r; prove that the sum of the reciprocals of the distances of the angular points of the polygon from a tangent to the circle is $\frac{n^2}{2r}\operatorname{cosec}^2\frac{n}{2}\theta$, where θ is the angle which a radius drawn to the point of contact of the tangent makes with the radius drawn to one of the angular points of the polygon.

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59. Find the first three terms of the series, in ascending powers of x, for $\tan x$.
Hence find the first three terms of the series for $\sec^2 x$.
Check by finding the series for $\sec x$ from the series for $\cos x$, and hence the series for $\sec^2 x$.

60. Prove that, when $x=\tan 2\theta$ and θ lies between $-\frac{\pi}{8}$ and $\frac{\pi}{8}$,

$$\tan\theta=\frac{x}{2}\left(1-\frac{x^2}{4}+\frac{x^4}{8}-\frac{5x^6}{64}+\ldots\right),$$

and that, if powers of x above the fifth are neglected,

$$\sin\theta=\frac{x}{2}\left(1-\tfrac{3}{8}x^2+\tfrac{31}{128}x^4\right).$$

61. Prove that, when x is real,

$$x+\frac{x^4}{\underline{|4}}+\frac{x^7}{\underline{|7}}+\ldots=\tfrac{1}{3}e^x+\tfrac{2}{3}e^{-\frac{1}{2}x}\sin\left(x\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right).$$

62. Sum the series $\quad x-\dfrac{x^7}{7}+\dfrac{x^{13}}{13}-\ldots. \quad (x<1)$

63. Prove that the coefficient of x^n in the expansion of $e^x\cos x$ in ascending powers of x is $\dfrac{2^{\frac{n}{2}}}{\underline{|n}}\cos\dfrac{n\pi}{4}$.

64. Sum $\quad \cos\dfrac{\theta}{2}+\dfrac{1}{2}\cos\dfrac{3\theta}{2}+\dfrac{1}{2^2}\cos\dfrac{5\theta}{2}+\ldots$ *ad infin.*

65. If $\quad a=\cos^2\theta-\frac{1}{3}\cos^3\theta\cos 3\theta+\frac{1}{5}\cos^5\theta\cos 5\theta-\ldots$ *ad infin.*,

prove that $\quad \tan 2a=2\cot^2\theta.$

66. Prove that θ is the sum of a constant and one of the two series

$$\tan\theta-\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta-\ldots,$$

or

$$-\cot\theta+\tfrac{1}{3}\cot^3\theta-\tfrac{1}{5}\cot^5\theta-\ldots,$$

distinguishing the cases.

67. If x is an acute angle and if $y=\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$, prove that

$$\cos x\cosh y=1,$$

and that $\quad \frac{1}{2}y=\sin x-\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x-\ldots.$

68. Sum the series (assuming them to be convergent):

(i) $\sinh u+\dfrac{1}{1.2}\sinh 2u+\dfrac{1}{1.2.3}\sinh 3u+\ldots,$

(ii) $\cosh u-\frac{1}{2}\cosh 2u+\frac{1}{3}\cosh 3u-\ldots.$

69. Sum to infinity the series

$$\cos\theta-\frac{1}{\underline{|3}}\cos 3\theta+\frac{1}{\underline{|5}}\cos 5\theta-\ldots.$$

70. Find the sum of the series

$$\frac{2}{1.3}\sin 2x-\frac{4}{3.5}\sin 4x+\frac{6}{5.7}\sin 6x-\ldots$$

for all values of x between 0 and π.

71. Prove that the coefficient of h^{n-1} in the expansion of $(1-xh+h^2)^{-1}$ is

$$(x-2\cos a)(x-2\cos 2a)(x-2\cos 3a)\ldots(x-2\cos\overline{n-1}\,a),$$

where $a=\dfrac{\pi}{n}$.

72. Expand $\dfrac{\sin\phi}{1-2x\cos\phi+x^2}$ in ascending powers of x, and prove that the remainder after n terms is $x^n\left(\dfrac{\sin\overline{n+1}\,\phi-x\sin n\phi}{1-2x\cos\phi+x^2}\right)$.

73. Prove that $\dfrac{1}{1+\sin 2\lambda\cos\theta}$ may be expanded into an infinite series

$$\sec 2\lambda\left[1+2\sum_{n=1}^{n=\infty}(-1)^n\tan^n\lambda\cos n\theta\right].$$

74. Show that

$$\log\left(1-\frac{2k}{1+k^2}\cos x\right)=-k^2+\tfrac{1}{2}k^4-\tfrac{1}{3}k^6+\dots$$
$$-2k\cos x-2\frac{k^2\cos 2x}{2}-2\frac{k^3\cos 3x}{3}-\dots.$$

75. If $\tan(\theta+\alpha)=\tan\alpha\sec 2\beta$,

prove that $\theta=\tan^2\beta\sin 2\alpha+\tfrac{1}{2}\tan^4\beta\sin 4\alpha+\tfrac{1}{3}\tan^6\beta\sin 6\alpha+\dots.$

76. Prove that

$$\frac{1}{\pi}=\frac{1}{4}\tan\frac{\pi}{4}+\frac{1}{8}\tan\frac{\pi}{8}+\frac{1}{16}\tan\frac{\pi}{16}+\dots\textit{ad infin.}$$

77. Sum the series $\displaystyle\sum_{r=1}^{r=\infty}\{(3r-1)^{-4}+(3r+1)^{-4}\}$.

78. Show that the sum of the rth powers of the first n odd integers, when r is a positive integer, is the coefficient of $\dfrac{x^r}{\lfloor r}$ in the expansion of $e^{nx}\dfrac{\sinh nx}{\sinh x}$.

79. Prove that, when $x^2<1$,

$$\log(x+\sqrt{1+x^2})+\sin^{-1}x=2\left\{x+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5.7}{2.4.6.8}\frac{x^9}{9}+\dots\right\}.$$

80. The roots of the equation $x^2-2bx+c=0$ are imaginary and an angle α is defined by the equation $b^2=c\cos^2\alpha$; prove that

$$\log(1-2by+cy^2)=-\sum_{n=1}^{n=\infty}\frac{2}{n}b^ny^n\sec^n\alpha\cos n\alpha.$$

81. If $y=\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)=a_1x+a_3x^3+a_5x^5+\dots,$

prove that $x=ay_1-a_3y^3+a_5y^5-\dots.$

82. Prove that the sum of the squares of the moduli of all the values of $(1+i)^{1+i}$ which are less than unity is equal to $e^{\frac{3\pi}{2}}\operatorname{cosech}2\pi$.

83. Show that $x^{10}+1=(x^2+1)\left(x^4-2x^2\cos\frac{\pi}{5}+1\right)\left(x^4-2x^2\cos\frac{3\pi}{5}+1\right)$. Hence find the numerical value of $\cos 36°$.

84. From the factors of $x^{2n}-2x^n\cos n\theta+1$, find the real quadratic factors of $\frac{x^{2n+1}+1}{x+1}$.

85. Find the value of $\sin\alpha\sin 3\alpha\sin 5\alpha\ldots\sin(2n-1)\alpha$, where $4n\alpha=\pi$.

86. Prove that, when n is odd, $\cot\frac{\pi}{2n}\cot\frac{3\pi}{2n}\cot\frac{5\pi}{2n}\ldots\cot\frac{\overline{n-2}\,\pi}{2n}=\sqrt{n}$, and find the value of $\cot\frac{\pi}{2n}\cot\frac{3\pi}{2n}\cot\frac{5\pi}{2n}\ldots\cot\frac{\overline{n-1}\,\pi}{2n}$, when n is even.

87. Find the value of $\cot\alpha\cot 3\alpha\cot 5\alpha\ldots\cot(2n+1)\alpha$, where $4(n+1)\alpha=\pi$.

88. Prove that

$$\sin n(\alpha-\beta)=(-1)^{n-1}\cos^n\alpha\sin n\left(\beta+\frac{\pi}{2}\right)\prod_{r=0}^{r=n-1}\left\{\tan\alpha-\tan\left(\beta+\frac{r\pi}{n}\right)\right\}.$$

89. Prove that

$$\prod_{n=-\infty}^{n=\infty}\left[1+\frac{x^3}{(n-a)^3}\right]=\frac{\sin\pi(a-x)\left\{\cosh(\pi x\sqrt{3})-\cos\pi(2a+x)\right\}}{2\sin^3\pi a}.$$

90. Taking the infinite product forms as definitions of $\sin\phi$ and $\cos\phi$, prove that they are periodic.

91. If n is odd, show that $\sin n\theta+\cos n\theta$ is divisible either by $\sin\theta+\cos\theta$ or by $\sin\theta-\cos\theta$.

92. Show that $\cos\frac{\pi}{10}=\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$ and $\cos\frac{3\pi}{10}=\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$.

93. Show that $4\cos^2\frac{\pi}{7}$ is a root of the equation $x^3-5x^2+6x-1=0$, and find the other roots.

94. Prove that the roots of the equation $x^3-21x^2+35x-7=0$ are

$$\tan^2\frac{\pi}{7},\quad \tan^2\frac{3\pi}{7},\quad \tan^2\frac{5\pi}{7}.$$

95. Express x^7-1 as the product of linear factors.

If $\theta=\frac{\pi}{7}$, show that the product of

$$\exp(2\theta i)+\exp(4\theta i)+\exp(8\theta i)\quad\text{and}\quad\exp(6\theta i)+\exp(10\theta i)+\exp(12\theta i)$$

is 2; and that $$\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\tfrac{1}{2}\sqrt{7}.$$

96. If $r=\cos\alpha+i\sin\alpha$, where $\alpha=\frac{2\pi}{7}$, show that $r+r^6$, r^2+r^5, r^3+r^4 are roots of a cubic with real integral coefficients.

97. Show how to find the equation $x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n=0$ which is satisfied by $x=\tan^2\frac{k\pi}{2n+1}$, where k is any integer.

Prove that the roots of $x^3+x^2-2x-1=0$ are $2\cos\frac{2\pi}{7}$, $2\cos\frac{4\pi}{7}$, $2\cos\frac{8\pi}{7}$.

98. Show that
$$\tan\frac{\pi}{9}+\tan\frac{4\pi}{9}+\tan\frac{7\pi}{9}=3\sqrt{3}.$$

99. Show that the equation whose roots are $\cos\frac{2\pi}{9}$, $\cos\frac{4\pi}{9}$, $\cos\frac{8\pi}{9}$ is $8x^3-6x+1=0$.

100. Prove that $\tan^2 20°$, $\tan^2 40°$, $\tan^2 80°$ are roots of the equation
$$x^3-33x^2+27x-3=0.$$

101. If $\alpha=\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{18\pi}{13}$, $\beta=\cos\frac{10\pi}{13}+\cos\frac{14\pi}{13}+\cos\frac{22\pi}{13}$, show that α, β are roots of the equation $4x^2+2x-3=0$. Hence evaluate α and β.

102. If $\alpha=\frac{\pi}{15}$, prove that
$$\cos 2\alpha+\cos 4\alpha+\cos 8\alpha+\cos 16\alpha=\tfrac{1}{2},$$
$$\sin 2\alpha+\sin 4\alpha+\sin 8\alpha+\sin 16\alpha=\frac{\sqrt{15}}{2}.$$

103. Prove that $2\cos 5\theta+1$ is divisible by $2\cos\theta+1$. Find the quotient and employ the result to show that $\sec^2 12°+\sec^2 24°+\sec^2 48°+\sec^2 96°=96$.

104. Prove that
$$\sum_{r=1}^{r=16}\operatorname{cosec}^2\frac{r\pi}{17}=96.$$

Sum to $(n+1)$ terms
$$\operatorname{cosec}^2\frac{\pi}{4(n+1)}+\operatorname{cosec}^2\frac{3\pi}{4(n+1)}+\operatorname{cosec}^2\frac{5\pi}{4(n+1)}+\dots$$

105. If $x=\cos 3\theta+\cos 5\theta+\cos 6\theta+\cos 7\theta$, where $\theta=\frac{2\pi}{17}$, prove that
$$x=-\frac{\sqrt{17}+1}{4}.$$

106. Find the relation between p, q, r if two values of θ satisfying the equation $p \sin \theta + q \cos \theta = r \sin \theta \cos \theta$ differ by $\frac{\pi}{2}$.

107. Show that the equation $\sin(\theta + \lambda) = a \sin 2\theta + b$ has four roots $\alpha, \beta, \gamma, \delta$ such that their sum is an odd multiple of two right angles.

108. Show that the equation $\cos(2\theta - \alpha) = k \cos(\theta - \beta)$ can be satisfied by four values, $\theta_1, \theta_2, \theta_3, \theta_4$, of θ, of which no two differ by a multiple of 2π, and by no more.

Show also that $$\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\alpha = 2r\pi.$$

109. Prove that, if the equation $a^2 \cos 2\theta \sec 2x - b^2 \sin 2\theta \operatorname{cosec} 2x = v^2 - b^2$ is satisfied by values α, β, γ such that no one of them differs from another, or from θ, by a multiple of π,
$$\tan(\alpha + \beta + \gamma) + \tan \alpha \tan \beta \tan \gamma = 0,$$
and that, if $b^2 = 2a^2$, $\quad \tan^3 \alpha = \tan^3 \beta = \tan^3 \gamma = -\cot \theta.$

110. If x_1, x_2, x_3, x_4 are the roots of the equation
$$x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos \beta - \sin \beta = 0,$$
prove that $\Sigma \tan^{-1} x_1 + \beta = n\pi + \frac{\pi}{2}$, where n is an integer.

111. Prove that the equation $\sin 3\theta = p \sin \theta + q \cos \theta + r$ has six roots, the sum of which is an odd multiple of π.

112. Show that, for six values of θ which satisfy the equation $\sin 3\theta = c$ and do not differ by multiples of π, $\Sigma \sin^2 \theta = 3$.

113. If $\tan \alpha$, $\tan \beta$, $\tan \gamma$ are all different and such that
$$\tan 3\alpha = \tan 3\beta = \tan 3\gamma,$$
then $$(\tan \alpha + \tan \beta + \tan \gamma)(\cot \alpha + \cot \beta + \cot \gamma) = 9.$$

Generalise the theorem to the case of $(2n+1)$ angles $\alpha, \beta, \ldots, \lambda$.

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114. If $u_r = (2 \cos \theta)^r - (r-1)(2 \cos \theta)^{r-2} + \frac{(r-2)(r-3)}{1 \cdot 2}(2 \cos \theta)^{r-4} - \ldots,$

prove that $$u_r - 2 \cos \theta \, u_{r-1} + u_{r-2} = 0.$$

115. Prove that the product of the n factors
$$(1 + \sec 2\alpha)(1 + \sec 2^2\alpha) \ldots (1 + \sec 2^n\alpha) \text{ is } \frac{\tan 2^n\alpha}{\tan \alpha}.$$

116. Prove that the product of the n factors
$$(2 \cos \alpha - 1)(2 \cos 2\alpha - 1)(2 \cos 2^2\alpha - 1) \ldots (2 \cos 2^{n-1}\alpha - 1) \text{ is } \frac{2 \cos 2^n\alpha + 1}{2 \cos \alpha + 1}.$$

117. Prove that the product of the n factors

$$\left(\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}\right)\left(\cos\frac{\alpha}{4}+\cos\frac{\beta}{4}\right)\left(\cos\frac{\alpha}{8}+\cos\frac{\beta}{8}\right)\dots\left(\cos\frac{\alpha}{2^n}+\cos\frac{\beta}{2^n}\right)$$

is

$$\frac{1}{2^n}\,\frac{\cos\alpha-\cos\beta}{\cos\frac{\alpha}{2^n}-\cos\frac{\beta}{2^n}}.$$

118. Prove, without forming an equation, that if $\alpha=\frac{\pi}{9}$,

$$16\cos\alpha\cos 2\alpha\cos 3\alpha\cos 4\alpha=1.$$

119. Prove that

$$16\sin\frac{\pi}{30}\sin\frac{7\pi}{30}\sin\frac{11\pi}{30}\sin\frac{17\pi}{30}=1.$$

120. If x, y, z are real and $A+B+C=\pi$, prove that

$$x^2+y^2+z^2>2yz\cos A+2zx\cos B+2xy\cos C,$$

unless

$$x:y:z=\sin A:\sin B:\sin C.$$

121. If $x^2+y^2+z^2-2yz\cos A-2zx\cos B-2xy\cos C=0$, then the only real values of x, y, z are given by $x:y:z=\sin A:\sin B:\sin C$.

122. Prove that in any triangle

$$c^2\cos^2 C\cos^2 2C=a^2\cos^2 A\cos^2 2A+b^2\cos^2 B\cos^2 2B$$
$$+2ab\cos A\cos B\cos 2A\cos 2B\cos 4C.$$

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123. If A, B, C are the angles of a triangle, show that

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}=0,$$

and, by expanding the determinant, prove that

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$$

[Use $a=b\cos C+c\cos B$.]

124. If A, B, C, D be the angles of a quadrilateral, prove that

$$\begin{vmatrix} 1 & -\cos A & \cos(A+B) & -\cos D \\ -\cos A & 1 & -\cos B & \cos(B+C) \\ \cos(A+B) & -\cos B & 1 & -\cos C \\ -\cos D & \cos(B+C) & -\cos C & 1 \end{vmatrix}=0.$$

125. If

$$\begin{vmatrix} 1 & \cos x & 0 & 0 \\ \cos x & 1 & \cos\alpha & \cos\beta \\ 0 & \cos\alpha & 1 & \cos\gamma \\ 0 & \cos\beta & \cos\gamma & 1 \end{vmatrix}=0,$$

show that

$$\sin^2\gamma\sin^2 x=\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta\cos\gamma.$$

126. Show that $\begin{vmatrix} 1 & \cos\theta & \cos 2\theta \\ \cos\theta & \cos 2\theta & \cos 3\theta \\ \cos 2\theta & \cos 3\theta & \cos 4\theta \end{vmatrix} = 0.$

127. Show that $\begin{vmatrix} 1 & \cos\theta & \cos 2\theta & \cos 3\theta \\ \cos\theta & \cos 2\theta & \cos 3\theta & \cos 4\theta \\ \cos 2\theta & \cos 3\theta & \cos 4\theta & \cos 5\theta \\ \cos 3\theta & \cos 4\theta & \cos 5\theta & \cos 6\theta \end{vmatrix} = 0.$

Show also that each of the first minors of this determinant is zero.

128. Show that

$$\begin{vmatrix} \sin(\alpha+x) & \sin(\alpha+y) & \sin(\alpha+z) \\ \sin(\beta+x) & \sin(\beta+y) & \sin(\beta+z) \\ \sin(\gamma+x) & \sin(\gamma+y) & \sin(\gamma+z) \end{vmatrix} = 0.$$

[Express as the product of two determinants.]

Show that, if $\alpha+\beta+\gamma+x+y+z=0$, the determinant with "tan" written for "sin" also vanishes.

129. If A, B, C are the angles of a triangle, prove that

$$\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = 0.$$

130. Prove that, if $\begin{vmatrix} 1 & -\cos z & -\cos y \\ -\cos z & 1 & -\cos x \\ -\cos y & -\cos x & 1 \end{vmatrix} = 0,$

then $$x \pm y \pm z = (2n+1)\pi.$$

131. If $A+B+C=\pi$, show that $\tan A$, $\tan B$ and $\tan C$ are roots of an equation $t^3 - pt^2 + qt - p = 0$. Hence show that

$$\begin{vmatrix} 1 & \cot^2 A & \cot A & \operatorname{cosec}^2 A \\ 1 & \cot^2 B & \cot B & \operatorname{cosec}^2 B \\ 1 & \cot^2 C & \cot C & \operatorname{cosec}^2 C \end{vmatrix} = 0.$$

132. Show that, if α, β, γ are all positive and each less than $\frac{\pi}{4}$, the determinant

$$\begin{vmatrix} \sin^2\alpha & \cot\alpha & 1 \\ \sin^2\beta & \cot\beta & 1 \\ \sin^2\gamma & \cot\gamma & 1 \end{vmatrix}$$

can vanish only if two of the angles are equal.

133. Show that

$$\begin{vmatrix} \sin^2 \alpha & \sin^2 2\alpha & \sin^2 3\alpha \\ \cos^2 \alpha & \cos^2 2\alpha & \cos^2 3\alpha \\ \cos^2 2\alpha & \cos^2 3\alpha & \cos^2 4\alpha \end{vmatrix} = -\sin^2\alpha \sin^2 2\alpha.$$

134. Prove that the sum of the two determinants

$$\begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \sin 2A & \sin 2B & \sin 2C \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 1 & 1 \\ \cos A & \cos B & \cos C \\ \cos 2A & \cos 2B & \cos 2C \end{vmatrix}$$

is $2\Sigma \sin A \Sigma \sin (B - C)$.

135. Show that

$$\begin{vmatrix} \sin 2\alpha & \cos \alpha & \sin \alpha \\ \sin 2\beta & \cos \beta & \sin \beta \\ \sin 2\gamma & \cos \gamma & \sin \gamma \end{vmatrix} = -4 \sin \frac{\beta-\gamma}{2} \sin \frac{\gamma-\alpha}{2} \sin \frac{\alpha-\beta}{2} \times \{\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta)\}.$$

136. If

$$\begin{vmatrix} \sin 2\alpha & \cos \alpha & \sin \alpha & 1 \\ \sin 2\beta & \cos \beta & \sin \beta & 1 \\ \sin 2\gamma & \cos \gamma & \sin \gamma & 1 \\ \sin 2\delta & \cos \delta & \sin \delta & 1 \end{vmatrix} = 0,$$

and if $\alpha, \beta, \gamma, \delta$ are different positive angles each less than 2π, show that

$$\alpha+\beta+\gamma+\delta = n\pi,$$

where n is an odd integer.

137. Prove that

$$\begin{vmatrix} \cos\theta & \sin\theta\cos\phi & \sin\theta\sin\phi\cos\psi & \sin\theta\sin\phi\sin\psi \\ -\sin\theta & \cos\theta\cos\phi & \cos\theta\sin\phi\cos\psi & \cos\theta\sin\phi\sin\psi \\ 0 & -\sin\theta\sin\phi & \sin\theta\cos\phi\cos\psi & \sin\theta\cos\phi\sin\psi \\ 0 & 0 & -\sin\theta\sin\phi\sin\psi & \sin\theta\sin\phi\cos\psi \end{vmatrix} = \sin^2\theta\sin\phi.$$

138. Prove that

$$\begin{vmatrix} \cos\theta & 1 & 0 & 0 & \cdots & \cdots \\ 1 & 2\cos\theta & 1 & 0 & \cdots & \cdots \\ 0 & 1 & 2\cos\theta & 1 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 & 2\cos\theta & 1 \\ \cdots & \cdots & \cdots & 0 & 1 & 2\cos\theta \end{vmatrix} = \cos n\theta,$$

if the determinant is of the nth order.

ANSWERS

EXERCISE XVII. a.

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292. 3. See p. 256 (iv) $\dfrac{3\tanh u+\tanh^3 u}{1+3\tanh^2 u}$.

293. 4. (i) $\sin x\cosh y+i\cos x\sinh y$. (ii) $\cos x\cosh y+i\sin x\sinh y$.
(iii) $\cosh x\cos y+i\sinh x\sin y$. (iv) $\sinh x\cos y+i\cosh x\sin y$.
(v) $\dfrac{2\cos x\cosh y+2i\sin x\sinh y}{\cos 2x+\cosh 2y}$. (vi) $\dfrac{2\sinh x\cos y-2i\cosh x\sin y}{\cosh 2x-\cos 2y}$.
(vii) $\dfrac{\sinh 2x+i\sin 2y}{\cosh 2x+\cos 2y}$.

EXERCISE XVII. b.

297. 10. $\sin^{-1}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)+i\cosh^{-1}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)$.

11. $\cos^{-1}\frac{1}{2}\{\sqrt{(1+\alpha)^2+\beta^2}-\sqrt{(1-\alpha)^2+\beta^2}\}$
$$\mp\cosh^{-1}\tfrac{1}{2}\{\sqrt{(1+\alpha)^2+\beta^2}+\sqrt{(1-\alpha)^2+\beta^2}\}$$
the upper or lower sign being taken according as β is positive or negative.

EXERCISE XVII. c.

301. 1. (i) $2n\pi i$. (ii) $\log a+i(2n+1)\pi$. (iii) $i(2n+\frac{1}{2})\pi$.

302. 8. $\frac{1}{2}\log\dfrac{\cosh 2y-\cos 2x}{2}+i\{2n\pi+\tan^{-1}(\cot x\tanh y)\}$,

if $\cot x\tanh y$ is positive;

or $\frac{1}{2}\log\dfrac{\cosh 2y-\cos 2x}{2}+i\{(2n+1)\pi+\tan^{-1}(\cot x\tanh y)\}$,

if $\cot x\tanh y$ is negative.

10. $(1, \frac{1}{2}\log\overline{x^2+y^2})\exp\left(2n\pi-\tan^{-1}\dfrac{y}{x}\right)$, when x is positive;

or $(1, \frac{1}{2}\log\overline{x^2+y^2})\exp\left(\overline{2n+1}\,\pi-\tan^{-1}\dfrac{y}{x}\right)$, when x is negative.

11. $\left(1, \dfrac{4n+1}{6}\pi\right)$, $\left(\sqrt{2\cos\dfrac{\overline{4n+1}\,\pi}{8}}, \dfrac{\overline{4n+1}\,\pi}{16}\right)$.

EXERCISE XIX. a.

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313. 1. (i) $\dfrac{\sin\theta}{\cos^2\theta}$. (ii) $\dfrac{4\cos\theta-2}{5-4\cos\theta}$. (iii) $\dfrac{6\sin\theta}{5-3\cos 2\theta}$.

(iv) $\dfrac{x\sin\theta}{1-2x\cos\theta+x^2}$. (v) $\dfrac{\cos\alpha-x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2}$.

(vi) $\dfrac{1-x\cosh\alpha}{1-2x\cosh\alpha+x^2}$, $x<e^{\alpha}$ and $<e^{-\alpha}$.

2. (i) $(1+2x\cos\theta+x^2)^{\frac{n}{2}}\cos n\left(\tan^{-1}\dfrac{x\sin\theta}{1+x\cos\theta}\right)$.

(ii) $\dfrac{1}{\sqrt{2\sin\frac{\theta}{2}}}\sin\left(\dfrac{\pi}{4}-\dfrac{\theta}{4}\right)$. (iii) $-1+\dfrac{1}{\left(2\sin\frac{\theta}{2}\right)^n}\cos\dfrac{n}{2}(\pi-\theta)$.

(iv) $\sqrt{2\cos\dfrac{\theta}{2}}\sin\dfrac{\theta}{4}$.

(v) $\dfrac{1}{\sqrt[3]{r}}\cos\dfrac{\phi}{3}$, where $r^2=1-2x\cos\theta+x^2$, $\phi=\tan^{-1}\dfrac{x\sin\theta}{1-x\cos\theta}$.

3. (i) $e^{-\cos x}\sin(\sin x)$. (ii) $e^{x\cos\alpha}\sin(x\sin\alpha)$.
(iii) $e^{-x\cos\theta}\cos(x\sin\theta)$. (iv) $e^{\cos\alpha}\cos(\theta+\sin\alpha)$.

(v) $\frac{1}{2}e^{\cos\frac{\beta}{2}}\cos\left(\alpha+\sin\dfrac{\beta}{2}\right)+\frac{1}{2}e^{-\cos\frac{\beta}{2}}\cos\left(\alpha-\sin\dfrac{\beta}{2}\right)$.

4. (i) $\dfrac{x}{2}$. (ii) $-\log\left(2\sin\dfrac{\theta}{2}\right)$. (iii) $\tan^{-1}\dfrac{a\sin x}{1+a\cos x}$.

(iv) (a) $\frac{1}{2}\log\left(\cot\dfrac{x}{2}\right)$, ($b$) $\frac{1}{2}\log\left(-\cot\dfrac{x}{2}\right)$.

314. 5. $\dfrac{x\sinh\alpha}{1-2x\cosh\alpha+x^2}$, $x<e^{\alpha}$ and $<e^{-\alpha}$. 6. 0. 7. $\dfrac{\cos^2\theta}{\sin\theta}$.

8. $\dfrac{1}{2-2\cot\theta+\cot^2\theta}$. 9. $e^{\cos^2\beta}\cos(\alpha+\sin\beta\cos\beta)$.

10. $(1-2x\cos 2\theta+x^2)^{-\frac{n}{2}}\sin\left(\theta+n\tan^{-1}\dfrac{x\sin 2\theta}{1-x\cos 2\theta}\right)$.

11. $\dfrac{\pi\sin x}{1+2\pi\cos x+\pi^2}$.

12. $\frac{1}{2}\sin(\alpha-\beta)\log(1+2x\cos\beta+x^2)+\cos(\alpha-\beta)\tan^{-1}\dfrac{x\sin\beta}{1+x\cos\beta}$.

13. $e^{x\sin\alpha\cos\alpha}\cos(x\sin^2\alpha)$. 14. $e^{\sin\theta}\cos(\tan\theta\sin\theta)$.

15. $\frac{1}{2}e^{\cos\beta}\cos(\alpha-\beta+\sin\beta)-\frac{1}{2}e^{-\cos\beta}\cos(\alpha-\beta-\sin\beta)$. 16. $\dfrac{\pi}{2}-\theta$.

17. $2\cos 2\theta$.

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314. 18. $(1+2x\cos\theta+x^2)^{\frac{n}{2}}\cos(\alpha+n\phi)$, where $\phi=\tan^{-1}\dfrac{x\sin\theta}{1+x\cos\theta}$.

19. $-\frac{1}{2}\log 2-\frac{1}{2}\log(\cos\alpha\sim\cos\beta)$. 20. $\frac{1}{3}+\dfrac{1+2\cos 2\theta}{5+4\cos 2\theta}$.

21. θ between $4n\pi$ and $(4n+2)\pi$.

EXERCISE XIX. b.

315. 1. $\sin^2 2\alpha$. 2. $\frac{1}{2}\operatorname{cosec} 2\alpha-\dfrac{1}{4\alpha}$. 3. $\frac{1}{4}(\alpha-\sin\alpha)$.

4. $\frac{1}{2}\sin 2\theta-\theta$. 5. $\tan^{-1}3$. 6. $\dfrac{\cos\theta-x}{1-2x\cos\theta+x^2}$.

7. $\dfrac{\cos\alpha-x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2}$. 8. $\dfrac{\sin\alpha-x\sin(\alpha-\beta)}{1-2x\cos\beta+x^2}$.

9. $\dfrac{(n+1)\cos n\theta-n\cos(n+1)\theta-1}{2(1-\cos\theta)}$. 10. $4\operatorname{cosec}^2 x-\dfrac{1}{x^2}$.

EXERCISE XIX. c.

316. 5. $\frac{1}{2}\tan^{-1}\dfrac{2a\sin\theta}{1-a^2}$; $\dfrac{\pi}{4}$; $-\dfrac{\pi}{4}$.

317. 10. $e^{\cos^2 y}\sin(x+\sin y\cos y)$.

12. $\frac{1}{2}\log(\pm\operatorname{cosec}\alpha)$, according as $\operatorname{cosec}\alpha$ is positive or negative.

13. $\sin^n y\sin\left(x+ny-\dfrac{n\pi}{2}\right)$.

14. $n\theta+2p\pi$, when $\dfrac{\sin(n+r)\theta}{\sin r\theta}$ is positive; or $n\theta+\pi+2\theta\pi$, when it is negative; p being chosen so that the angle is between $\pm\pi$.

15. $\sin(\sin\theta\cos\theta)\cosh(\cos^2\theta)$.

17. (i) $1-2\sin^2\dfrac{\theta}{2}\log\left(2\sin\dfrac{\theta}{2}\right)-\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)\sin\theta$.

(ii) $\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)2\sin^2\dfrac{\theta}{2}-\sin\theta\log\left(2\sin\dfrac{\theta}{2}\right)$.

318. 22. $\frac{1}{2}\cosh(\cos\phi)\cos(\sin\phi)-\frac{1}{2}\cos(\cos\phi)\cosh(\sin\phi)$.

24. $\frac{1}{2}\sinh(\cos^2\alpha)\cos(\sin\alpha\cos\alpha)-\frac{1}{2}\sin(\cos^2\alpha)\cosh(\sin\alpha\cos\alpha)$.

EXERCISE XXI. a.

332. 1. (i) 0·0998, 0·9950. (ii) 0·4794, 0·8776.

2. 4° 26′. 3. 1° 59′. 4. 1° 9′. 5. 88° 51·2 (4)′.

6. $\dfrac{\pi}{2}-x-\dfrac{1}{2}\dfrac{x^3}{3}-\dfrac{1.3}{2.4}\dfrac{x^5}{5}-\ldots$. 7. $x-\dfrac{1}{2}\dfrac{x^3}{3}+\dfrac{1.3}{2.4}\dfrac{x^5}{5}-\ldots$.

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333. 13. $\frac{\theta^7}{1050}$. 15. $a = -\frac{1}{6}$, $b = -\frac{4}{3}$.

334. 19. (i) 1. (ii) 3. (iii) $\frac{2(b^2 - a^2)}{c^2}$. (iv) 0. (v) $e^{-\frac{1}{2}a^2}$.

20. -1. 21. 0.

EXERCISE XXI. b.

335. 1. $\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \dots$. 5. $\sqrt{3}\,\frac{\pi}{6}$.

336. 7. $\frac{1}{2}\tan^{-1}\frac{2x\cos\theta}{1-x^2}$.

EXERCISE XXI. c.

339. 1. $-2\left[x\cos\theta + \frac{1}{2}x^2\cos 2\theta + \frac{1}{3}x^3\cos 3\theta + \dots\right]$.

2. (i) $1 - x\cos\theta + x^2\cos 2\theta - \dots$.

(ii) $2\{x\cos\theta + x^2\sin 2\theta - x^3\cos 3\theta - x^4\sin 4\theta + \dots\}$.

(iii) $1 + 2x\cos\theta + 2x^2\cos 2\theta + \dots$.

(iv) $1 + x\frac{\sin 2\theta}{\sin\theta} + x^2\frac{\sin 3\theta}{\sin\theta} + \dots$.

4. (i) $1 + \frac{x\cos\theta}{\lfloor 1} + \frac{x^2\cos 2\theta}{\lfloor 2} + \dots$.

(ii) $1 + \frac{xr\cos\theta}{\lfloor 1} + \frac{x^2r^2\cos 2\theta}{\lfloor 2} + \dots$, where $a + ib = (r, \theta)$.

5. $\sin\theta + \frac{1}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + \dots$.

6. $2\left(-\tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} - \frac{1}{5}\tan^{10}\frac{\theta}{2} - \dots\right)$.

7. $-\frac{1}{2}\tan^2\theta + \frac{1}{4}\tan^4\theta - \frac{1}{6}\tan^6\theta + \dots$.

11. $r\sin\theta + \frac{1}{2}r^2\sin 2\theta + \frac{1}{3}r^3\sin 3\theta + \dots$.

12. $y\cos\alpha - \frac{1}{2}y^2\sin 2\alpha - \frac{1}{3}y^3\cos 3\alpha + \frac{1}{4}y^4\sin 4\alpha + \dots$.

340. 20. $1 + \frac{x2^{\frac{1}{2}}\cos\frac{\pi}{4}}{\lfloor 1} + \frac{x^2 2^{\frac{2}{2}}\cos\frac{2\pi}{4}}{\lfloor 2} + \frac{x^3 2^{\frac{3}{2}}\cos\frac{3\pi}{4}}{\lfloor 3} + \dots$.

EXERCISE XXII. b.

356. 1. (i) See § 4. (ii) See § 8.

2. (i)—(iv) See § 7. (v) $\prod_{p=0}^{p=2}\left[x^2 - 2x\cos\left(\frac{4\pi}{9} + \frac{2p\pi}{3}\right) + 1\right]$.

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356. 3. (*a*) (i) (ii) See § 3. (iii) (iv) See § 2.

(v) $2^3 \prod_{p=0}^{p=3}\left[\cosh u - \cos\left(\frac{\pi}{8}+\frac{p\pi}{2}\right)\right]$. (vi) $2^4 \prod_{p=1}^{p=4}\left(\cosh u - \cos\frac{p\pi}{5}\right)$.

(*b*) (i) (ii) See § 11. (iii) (iv) See § 10.

(v) $\left(1+\dfrac{\sinh^2 u}{\sin^2\frac{\pi}{8}}\right)\left(1+\dfrac{\sinh^2 u}{\sin^2\frac{3\pi}{8}}\right)$.

(vi) $5\left(1+\dfrac{\sinh^2 u}{\sin^2\frac{\pi}{5}}\right)\left(1+\dfrac{\sinh^2 u}{\sin^2\frac{2\pi}{5}}\right)$.

4. (i) $\prod_{p=0}^{p=2}\left[x^2-2x\cos\left(\frac{\pi}{9}+\frac{2p\pi}{3}\right)+1\right]$.

(ii) $\prod_{p=0}^{p=5}\left[x^2-2x\cos\left(\frac{\pi}{18}+\frac{p\pi}{3}\right)+1\right]$.

(iii) $\prod_{p=0}^{p=4}\left[x^2-2x\cos\left(\frac{2\pi}{15}+\frac{2p\pi}{5}\right)+1\right]$. 5. See §§ 5, 6.

6. (i) $4(\sin\alpha-\sin\beta)\left(\sin\alpha-\sin\overline{\frac{\pi}{3}-\beta}\right)\left(\sin\alpha-\sin\overline{\frac{2\pi}{3}+\beta}\right)$.

(ii) $8\prod_{p=0}^{p=3}\left[\cos\left(\frac{\pi}{8}-\alpha\right)-\cos\left(\frac{\pi}{8}-\beta+\frac{p\pi}{2}\right)\right]$.

7. (i) $(x^2-a^2)\prod_{p=1}^{p=n-1}\left[x^2-2ax\cos\frac{p\pi}{n}+a^2\right]$.

(ii) $(x-a)\prod_{p=1}^{p=n}\left[x^2-2ax\cos\frac{2p\pi}{2n+1}+a^2\right]$.

(iii) $\prod_{p=0}^{p=n-1}\left[x^2-2ax\cos\left(\frac{\pi}{2n}+\frac{2p\pi}{n}\right)+a^2\right]$.

(iv) $(x+a)\prod_{p=1}^{p=n}\left[x^2-2ax\cos\frac{(2p+1)\pi}{2n+1}+a^2\right]$.

357. 12. (ii) $\cos n\theta$. (v) $2^{\frac{p-1}{2}}\sin\frac{\pi}{2p}\sin\frac{3\pi}{2p}\dots\sin\frac{\overline{p-2}\,\pi}{2p}=1$.

13. $2^{n-1}\cos\frac{\pi}{2n}\cos\frac{5\pi}{2n}\dots\cos\frac{\overline{4n-3}\,\pi}{2n}=(-1)^n\cos\frac{n\pi}{2}$.

EXERCISE XXII. c.

360. 16. $\dfrac{1}{5(x+1)}+\dfrac{2\left(1-x\cos\frac{\pi}{5}\right)}{5\left(x^2-2x\cos\frac{\pi}{5}+1\right)}+\dfrac{2\left(1+x\cos\frac{2\pi}{5}\right)}{5\left(x^2+2x\cos\frac{2\pi}{5}+1\right)}$.

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361. 17. (i) $$\sum_{p=0}^{p=n-1} \frac{x\cos\left(\overline{n-1}\,\theta - \frac{2p\pi}{n}\right) - a\cos n\theta}{na^{n-1}\left\{x^2 - 2ax\cos\left(\theta + \frac{2p\pi}{n}\right) + a^2\right\}}.$$

(ii) $$\sum_{p=0}^{p=n-1} \frac{x - a\cos\left(\theta + \frac{2p\pi}{n}\right)}{x^2 - 2ax\cos\left(\theta + \frac{2p\pi}{n}\right) + a^2}.$$

24. $$A = \frac{1}{10\left(\tan^2\frac{3\pi}{10} - \tan^2\frac{\pi}{10}\right)} = -B.$$

EXERCISE XXIII. c.

375. 1. $\frac{1}{4}\operatorname{cosec}^2\frac{\phi}{2}$. 3. $\frac{\pi}{8}\coth\frac{\pi}{4} - \frac{1}{2}$.

376. 8. $\frac{\pi^6}{945}$. 12. $\frac{\pi^2}{12}$. 13. $\frac{\pi^2}{6} + \frac{\pi^4}{90}$. 16. $\frac{\pi^2}{3} - 3$.

EXERCISE XXIV. a.

379. 1. $2{\cdot}196,\ 2\cosh\left(0{\cdot}439 + \frac{2\pi i}{3}\right),\ 2\cosh\left(0{\cdot}439 + \frac{4\pi i}{3}\right)$.

2. $1{\cdot}522,\ \frac{2\sqrt{3}}{3}\cosh\left(0{\cdot}777 + \frac{2\pi i}{3}\right),\ \frac{2\sqrt{3}}{3}\cosh\left(0{\cdot}777 + \frac{4\pi i}{3}\right)$.

3. $0{\cdot}596,\ 2\sinh\left(0{\cdot}294 + \frac{2\pi i}{3}\right),\ 2\sinh\left(0{\cdot}294 + \frac{4\pi i}{3}\right)$.

4. $-1{\cdot}431,\ -\frac{2\sqrt{3}}{3}\cosh\left(0{\cdot}679 + \frac{2\pi i}{3}\right),\ -\frac{2\sqrt{3}}{3}\cosh\left(0{\cdot}679 + \frac{4\pi i}{3}\right)$.

EXERCISE XXIV. d.

388. 3. $t^6 - 21t^4 + 35t^2 - 7 = 0$. $\tan\frac{2\pi}{7}, \tan\frac{3\pi}{7}, \ldots, \tan\frac{6\pi}{7}$.

4. $1 - 24\sin^2\theta + 80\sin^4\theta - 64\sin^6\theta$. 5. $t^6 - 33t^4 + 27t^2 - 3 = 0$.

389. 7. $2\cos\frac{p\pi}{9}$, $p = 2, 4, 5, 7, 8$. $64z^3 - 96z^2 + 36z - 1 = 0$. 13. 15, 60.

MISCELLANEOUS EXERCISES IV

391. 2. (i) 0. (ii) $\frac{\pi}{2}$. 4. $u = (1, \pm\theta)$.

8. $\pm\left(\sqrt{2\cos\frac{\theta}{2}}\cos\frac{\theta}{4} + i\sqrt{2\cos\frac{\theta}{2}}\sin\frac{\theta}{4}\right)$, if $\cos\frac{\theta}{2}$ is $+$ve;

$\pm\left(\sqrt{-2\cos\frac{\theta}{2}}\sin\frac{\theta}{4} - i\sqrt{-2\cos\frac{\theta}{2}}\cos\frac{\theta}{4}\right)$, if $\cos\frac{\theta}{2}$ is $-$ve.

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391. 9. $A=\pm\cosh^{-1}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)$, $B=2n\pi\pm\cos^{-1}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)$.

392. 15. $\frac{x^2}{c^2\cosh^2\phi}+\frac{y^2}{c^2\sinh^2\phi}=1$; $\frac{x^2}{c^2\cos^2\psi}-\frac{y^2}{c^2\sin^2\psi}=1$.

16. As x changes from 0 to $\sin^{-1}\frac{1}{\kappa}$, η is zero and ξ increases from 0 to $\frac{\pi}{2}$; while x changes from $\sin^{-1}\frac{1}{\kappa}$ to $\pi-\sin^{-1}\frac{1}{\kappa}$, $\xi=\frac{\pi}{2}$ and $\eta=\cosh^{-1}(\kappa\sin x)$; as x changes from $\pi-\sin^{-1}\frac{1}{\kappa}$ to π, η is zero and ξ decreases from $\frac{\pi}{2}$ to 0.

17. $e^{-\frac{\pi}{2}}(e^{-2\pi})^n$. 20. $-\frac{1}{2}-i\,\frac{1}{2}\cot\frac{p\pi}{n}$, $p=1, 2, 3, \ldots \overline{p-1}$.

393. 22. $\tan(x-\tan^{-1}x)$.

26. $[32\cos^5\theta-16\cos^4\theta-32\cos^3\theta+12\cos^2\theta+6\cos\theta-1]^2$.

27. $\frac{1}{2}\{\operatorname{cosec} x-\operatorname{cosec} 3^n x\}$. 28. $\tan^{-1}\frac{3n}{n+5}$.

29. $2^{2n}\operatorname{cosec}^2 2^n\theta-\operatorname{cosec}^2\theta-\frac{4^n-1}{3}$.

30. $\frac{1}{2}\sinh\theta\left(\coth\frac{\theta}{2^{n+1}}-\coth\frac{\theta}{2}\right)$. 31. $2^n\sin^n\frac{b}{2}\cos\left(a+\frac{nb}{2}-\frac{n\pi}{2}\right)$.

394. 35. (i) $2^n\cosh^n\frac{u}{2}\cosh\frac{nu}{2}-1$. (ii) $2^n\cosh^n\frac{u}{2}\sinh\frac{(n+2)u}{2}$.

37. $\frac{\sin^2 n\theta/2}{2\sin^2\theta/2}$.

40. $(-1)^{n+1}\,\frac{1}{4}\sec^2\frac{\alpha}{2}\left\{(n+1)^2\cos n\alpha+n^2\cos(n+1)\alpha-\cos(n+\frac{1}{2})\alpha\sec\frac{\alpha}{2}\right\}$.

41. $-\frac{\cos\pi/2n}{2\sin^3\pi/2n}+\frac{n(n+1)}{2}\cot\frac{\pi}{2n}$.

395. 43. $\log(x^{2n}-2x^n\cos\theta+1)$. 44. $\log\frac{\cos(n+1)\theta+\sin n\theta}{\cos(n+1)\theta-\sin n\theta}$.

46. $1-{}_nC_2\tan^2\theta+{}_nC_4\tan^4\theta-\ldots$.

47. $c_n\lambda^n+{}_{n-1}C_1\,c_{n-1}\lambda^{n-2}+{}_{n-2}C_2\,c_{n-2}\lambda^{n-4}+\ldots$.

49. $\frac{1}{4}\{\sin n\alpha\sin(n+3)\alpha\operatorname{cosec}\alpha+\sin 2\alpha-\sin(n+1)(n+2)\alpha\}$.

50. $\frac{\cos\theta\sin n\theta\sin(n+3)\theta}{4\sin^3\theta\sin(n+1)\theta\sin(n+2)\theta}$.

51. $\frac{1}{4}\left\{\sec^2\frac{n\alpha}{2}+\sec^2\frac{n+1}{2}\alpha-\sec^2\frac{\alpha}{2}-1\right\}$.

396. 59. $\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5$, $\sec^2 x=1+x^2+\frac{2}{3}x^4$.

397. 62. $\frac{1}{3}\{\tan^{-1}x+\omega^2\tan^{-1}\omega x+\omega\tan^{-1}\omega^2 x\}$, where $\omega^3=1$.

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397. **64.** $\dfrac{2\cos\theta/2}{5-4\cos\theta}$.

68. (i) $e^{\cosh u}\sinh(\sinh u)$. (ii) $\log\left(2\cosh\dfrac{u}{2}\right)$.

69. $\sin(\cos\theta)\cosh(\sin\theta)$.

70. $\mp\dfrac{\pi}{4}\cos x$, according as $\cos x$ is +ve or −ve.

398. **77.** $-1+\dfrac{8}{729}\pi^4$.

399. **83.** $\dfrac{\sqrt{5}+1}{4}$. **84.** See Chap. XXII, § 6, p. 345. **85.** $\dfrac{1}{2^{n-\frac{1}{2}}}$.

86. 1. **87.** 1.

93. $4\cos^2\dfrac{2\pi}{7}$, $4\cos^2\dfrac{3\pi}{7}$.

400. **96.** $x^3+x^2-2x+1=0$. **101.** $\alpha=\dfrac{\sqrt{13}-1}{4}$, $\beta=\dfrac{-\sqrt{13}-1}{4}$

103. $16\cos^4\theta-8\cos^3\theta-16\cos^2\theta+8\cos\theta+1$.

401. **106.** $2(p^2+q^2)^3=r^2(p^2-q^2)^2$.

113. $\Sigma\tan\alpha\,\Sigma\cot\alpha=(2n+1)^2$.

sinh x

x	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
0·0	0·0000	0·0100	0·0200	0·0300	0·0400	0·0500	0·0600	0·0701	0·0801	0·0901
0·1	0·1002	0·1102	0·1203	0·1304	0·1405	0·1506	0·1607	0·1708	0·1810	0·1911
0·2	0·2013	0·2115	0·2218	0·2320	0·2423	0·2526	0·2629	0·2733	0·2837	0·2941
0·3	0·3045	0·3150	0·3255	0·3360	0·3466	0·3572	0·3678	0·3785	0·3892	0·4000
0·4	0·4108	0·4216	0·4325	0·4434	0·4543	0·4653	0·4764	0·4875	0·4986	0·5098
0·5	0·5211	0·5324	0·5438	0·5552	0·5666	0·5782	0·5897	0·6014	0·6131	0·6248
0·6	0·6367	0·6485	0·6605	0·6725	0·6846	0·6967	0·7090	0·7213	0·7336	0·7461
0·7	0·7586	0·7712	0·7838	0·7966	0·8094	0·8223	0·8353	0·8484	0·8615	0·8748
0·8	0·8881	0·9015	0·9150	0·9286	0·9423	0·9561	0·9700	0·9840	0·9981	1·0122
0·9	1·0265	1·0409	1·0554	1·0700	1·0847	1·0995	1·1144	1·1294	1·1446	1·1598
1·0	1·1752	1·1907	1·2063	1·2220	1·2379	1·2539	1·2700	1·2862	1·3025	1·3190
1·1	1·3356	1·3524	1·3693	1·3863	1·4035	1·4208	1·4382	1·4558	1·4735	1·4914
1·2	1·5095	1·5276	1·5460	1·5645	1·5831	1·6019	1·6209	1·6400	1·6593	1·6788
1·3	1·6984	1·7182	1·7381	1·7583	1·7786	1·7991	1·8198	1·8406	1·8617	1·8829
1·4	1·9043	1·9259	1·9477	1·9697	1·9919	2·0143	2·0369	2·0597	2·0827	2·1059
1·5	2·1293	2·1529	2·1768	2·2008	2·2251	2·2496	2·2743	2·2993	2·3245	2·3499
1·6	2·3756	2·4015	2·4276	2·4540	2·4806	2·5075	2·5346	2·5620	2·5896	2·6175
1·7	2·6456	2·6740	2·7027	2·7317	2·7609	2·7904	2·8202	2·8503	2·8806	2·9112
1·8	2·9422	2·9734	3·0049	3·0367	3·0689	3·1013	3·1340	3·1671	3·2005	3·2341
1·9	3·2682	3·3025	3·3372	3·3722	3·4075	3·4432	3·4792	3·5156	3·5523	3·5894
2·0	3·6269	3·6647	3·7028	3·7413	3·7803	3·8196	3·8593	3·8993	3·9398	3·9806
2·1	4·0219	4·0635	4·1056	4·1480	4·1909	4·2342	4·2779	4·3221	4·3666	4·4116
2·2	4·4571	4·5030	4·5494	4·5962	4·6434	4·6913	4·7394	4·7881	4·8372	4·8868
2·3	4·9370	4·9876	5·0387	5·0903	5·1425	5·1951	5·2483	5·3020	5·3562	5·4109
2·4	5·4662	5·5221	5·5785	5·6354	5·6929	5·7510	5·8097	5·8689	5·9288	5·9892
2·5	6·0502	6·1118	6·1741	6·2369	6·3004	6·3645	6·4293	6·4946	6·5607	6·6274
2·6	6·6947	6·7628	6·8315	6·9008	6·9709	7·0417	7·1132	7·1854	7·2583	7·3319
2·7	7·4063	7·4814	7·5572	7·6338	7·7112	7·7894	7·8683	7·9480	8·0285	8·1098
2·8	8·1919	8·2749	8·3586	8·4432	8·5287	8·6150	8·7021	8·7902	8·8791	8·9689
2·9	9·0596	9·1512	9·2437	9·3371	9·4315	9·5268	9·6231	9·7203	9·8185	9·9177
3·0	10·018	10·119	10·270	10·324	10·429	10·534	10·640	10·748	10·856	10·966
3·1	11·076	11·188	11·301	11·415	11·530	11·647	11·764	11·883	12·003	12·124
3·2	12·246	12·369	12·494	12·620	12·747	12·876	13·006	13·137	13·269	13·403
3·3	13·538	13·674	13·812	13·951	14·092	14·234	14·377	14·522	14·668	14·816
3·4	14·965	15·116	15·268	15·422	15·577	15·734	15·893	16·053	16·214	16·378
3·5	16·543	16·709	16·877	17·047	17·219	17·392	17·567	17·744	17·923	18·103
3·6	18·285	18·469	18·655	18·843	19·033	19·224	19·418	19·613	19·811	20·010
3·7	20·211	20·415	20·620	20·828	21·037	21·249	21·463	21·679	21·897	22·117
3·8	22·339	22·564	22·791	23·020	23·252	23·486	23·722	23·961	24·202	24·445
3·9	24·691	24·939	25·190	25·444	25·700	25·958	26·219	26·483	26·749	27·018
4·0	27·290									
x	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09

cosh x

x	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
0·0	1·0000	1·0001	1·0002	1·0005	1·0008	1·0013	1·0018	1·0025	1·0032	1·0041
0·1	1·0050	1·0061	1·0072	1·0085	1·0098	1·0113	1·0128	1·0145	1·0162	1·0181
0·2	1·0201	1·0221	1·0243	1·0266	1·0289	1·0314	1·0340	1·0367	1·0395	1·0423
0·3	1·0453	1·0484	1·0516	1·0549	1·0584	1·0619	1·0655	1·0692	1·0731	1·0770
0·4	1·0811	1·0852	1·0895	1·0939	1·0984	1·1030	1·1077	1·1125	1·1174	1·1225
0·5	1·1276	1·1329	1·1383	1·1438	1·1494	1·1551	1·1609	1·1669	1·1730	1·1792
0·6	1·1855	1·1919	1·1984	1·2051	1·2119	1·2188	1·2258	1·2330	1·2402	1·2476
0·7	1·2552	1·2628	1·2706	1·2785	1·2865	1·2947	1·3030	1·3114	1·3199	1·3286
0·8	1·3374	1·3464	1·3555	1·3647	1·3740	1·3835	1·3932	1·4029	1·4128	1·4229
0·9	1·4331	1·4434	1·4539	1·4645	1·4753	1·4862	1·4973	1·5085	1·5199	1·5314
1·0	1·5431	1·5549	1·5669	1·5790	1·5913	1·6038	1·6164	1·6292	1·6421	1·6552
1·1	1·6685	1·6820	1·6956	1·7093	1·7233	1·7374	1·7517	1·7662	1·7808	1·7957
1·2	1·8107	1·8258	1·8413	1·8568	1·8725	1·8884	1·9045	1·9208	1·9373	1·9540
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1·5	2·3524	2·3738	2·3955	2·4174	2·4395	2·4619	2·4845	2·5073	2·5305	2·5538
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4·0	27·308									
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